How vortex bound states affect the Hall conductivity of a chiral p-wave superconductor

Daniel Ariad (daniel@ariad.org), Yshai Avishai and Eytan Grosfeld

Ben-Gurion University of the Negev

We represent a method to diagonalize a superconducting Hamiltonian in the presence of a vortex lattice, that employs only smooth gauge transformations. It renders the Hamiltonian to be periodic and enables the treatment of vortices of finite radii. The charge response c_{xy}, which is proportional to the Hall conductivity, is calculated using the Streda formula. The results reveal a quantized contribution to c_{xy} due to the formation of bound states, proportional to the system's Chern number.

The physical system

- Our setup consists vortex superlattice imposed on top of the electronic lattice of a chiral $p_x \pm i p_y$ superconductor.[1]

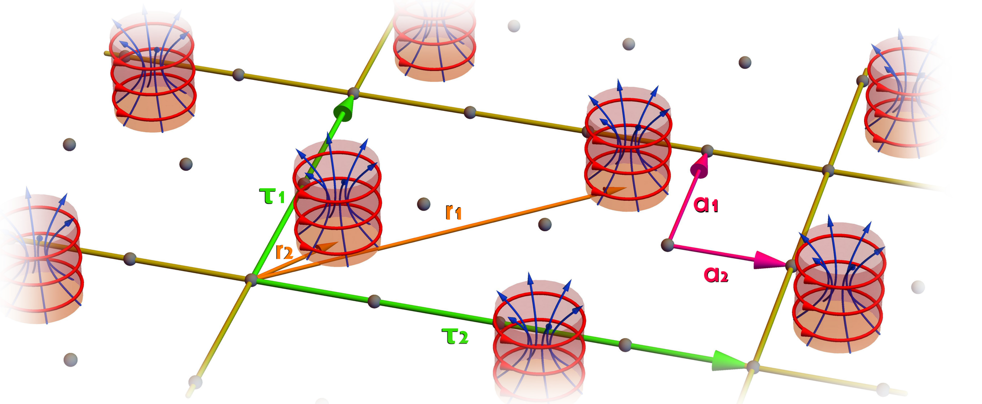


Figure 1: A magnetic field penetrating through vortices is depicted by blue field lines. Red field lines represent screening currents. $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ span the magnetic unit cell. $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are positions of the vortices.

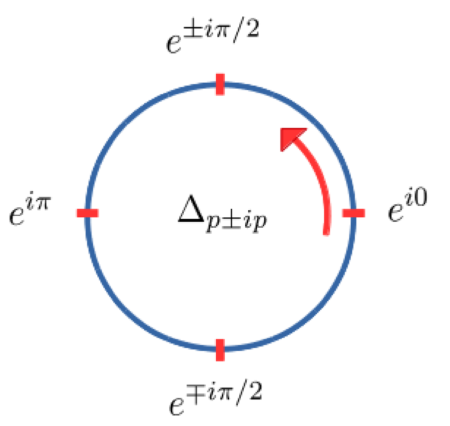
- The supercurrents in system are described by

$$\mathbf{J} = \frac{\rho_0}{2m_e}\left(\nabla\Theta - 2\mathbf{A}\right) - \frac{1}{4m_e}\left(\hat{z}\times\nabla\right)\left[\rho + c_{xy}\nabla\times\left(\nabla\Theta - 2\mathbf{A}\right)\right], \qquad (1)$$

where $\mathbf{A}, \Theta, m_e, \rho, \rho_0$ and c_{xy} are the vector potential, superconducting phase, electron mass, charge density, equilibrium density and charge response.

Hamiltonian and Order parameter

- The $p_x \pm i p_y$ BdG Hamiltonian in the tight-binding approximation (taking $\hbar = c = e = 1$) consists of three term, $\hat{H} = \hat{T} + \hat{\Delta} - \mu\hat{N}$ with



$$\hat{T} = -t\sum_{\boldsymbol{r},s,i} e^{i\int_{\boldsymbol{r}}^{\boldsymbol{r}+\boldsymbol{a}_i}\mathbf{A}\cdot d\boldsymbol{\ell}}\psi^\dagger_{\boldsymbol{r}+\boldsymbol{a}_i,s}\psi_{\boldsymbol{r},s} + \text{h.c.},$$

$$\hat{\Delta}_{p\text{-wave}} = \sum_{\boldsymbol{r},i}\Delta_{p\pm ip}(\boldsymbol{r},\boldsymbol{a}_i)\psi^\dagger_{\boldsymbol{r}\downarrow}\psi^\dagger_{\boldsymbol{r}+\boldsymbol{a}_i\downarrow} + \text{h.c.}, \qquad (2)$$

$$\Delta_{p\pm ip}(\boldsymbol{r},\boldsymbol{a}) = \Delta_0(\boldsymbol{r})e^{\pm i\text{Arg}(\boldsymbol{a})}e^{i\Theta(\boldsymbol{r})}e^{\frac{i}{2}\int_{\mathbf{r}}^{\mathbf{r}+\mathbf{a}}\nabla\Theta\cdot d\boldsymbol{\ell}}.$$

- $\hat{T}$, $\hat{\Delta}$ and $\mu\hat{N}$ are the hopping, coupling and on-site energy terms.
- $\psi^\dagger_{\boldsymbol{r}}$ and $\psi_{\boldsymbol{r}}$ are fermionic creation and annihilation operators.
- t, Δ_0 and μ are the hopping amplitude, order-parameter magnitude and chemical potential.
- $\boldsymbol{r}$, $s = \pm\uparrow\downarrow$ and $\boldsymbol{a}_i = a_i\hat{\boldsymbol{\tau}}_i$ with $i = 1, 2$ are the lattice vector, spin projection and primitive vectors.
- The factor $e^{\pm i\text{Arg}(\boldsymbol{a})}$ with $\text{Arg}(\boldsymbol{r}) \equiv \text{Arg}(x+iy)$ is due to the $p\pm ip$ symmetry of the gap and the superconducting gap is plotted above.
- $e^{i\int_{\boldsymbol{r}}^{\boldsymbol{r}+\boldsymbol{a}_i}\mathbf{A}\cdot d\boldsymbol{\ell}}$ is the Peierls phase factor and $e^{i\Theta(\boldsymbol{r})}e^{\frac{i}{2}\int_{\mathbf{r}}^{\mathbf{r}+\mathbf{a}}\nabla\Theta\cdot d\boldsymbol{\ell}}$ is a phase factor which encodes the superconductor response.

Vortex superlattice

- In the presence of of a vortex lattice, the superconducting phase is $\Theta(\boldsymbol{r}) = \sum_{i=1}^{N_v} s_i\theta(\boldsymbol{r}-\boldsymbol{r}_i)$, where N_V is the number of vortices per magnetic unit cell, $s_i = \pm 1$ is the winding number of the ith vortex and

$$\theta(\boldsymbol{r}-\boldsymbol{r}_i) = \lim_{M\to\infty}\left[\sum_{m,n=-2M}^{2M}\text{Arg}(\boldsymbol{r}-\boldsymbol{r}_i - m\boldsymbol{\tau}_1 - n\boldsymbol{\tau}_2) \mod 2\pi\right]. \qquad (3)$$

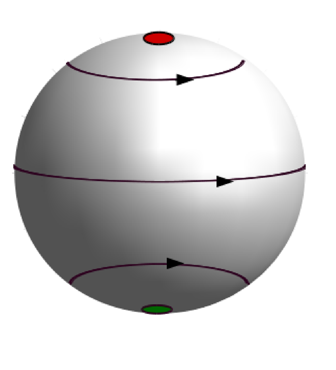
- We calculated it analytically and found that

$$\theta(z) = \text{Im}\left[\log\left(i\vartheta_1\left(\frac{z}{\tau_2}, -\frac{\tau_1}{\tau_2}\right)\right) - \frac{2iz^2}{\tau_1\tau_2}\text{arctg}\left(\frac{i\tau_1}{\tau_2}\right)\right], \qquad (4)$$

revealing that it is generally non-periodic on the magnetic unit cell.

Smooth gauge

- Stokes' theorem implies that on a compact geometry only vortex-antivortex pairs are allowed. We demonstrate it below for a sphere.
- We seek a gauge that renders the superconducting phase periodic only at the atomic lattice sites, $\mathbf{A} \to \mathbf{A} + \frac{1}{2}\nabla_{\boldsymbol{r}}\chi, \Delta \to \Delta e^{i\chi}, \psi_{\boldsymbol{r}s} \to e^{i\chi/2}\psi_{\boldsymbol{r}s}$.
- If $\mathbf{J} \propto \frac{1}{2}\nabla_{\boldsymbol{r}}\Theta - \mathbf{A}$ is doubly periodic than $\int_{\boldsymbol{r}}^{\boldsymbol{r}+\boldsymbol{\tau}_i}\mathbf{J}\cdot d\boldsymbol{\ell}$ is also. Therefore, we can choose $\chi(\boldsymbol{r})$ so that, $\Theta'(\boldsymbol{r}) = \Theta(\boldsymbol{r}) + \chi(\boldsymbol{r})$ and $\int_{\boldsymbol{r}}^{\boldsymbol{r}+\boldsymbol{\tau}_i}\left(\mathbf{A} + \frac{1}{2}\nabla_{\boldsymbol{r}}\chi\right)\cdot d\boldsymbol{\ell}$ are simultaneously periodic (mod 2π).
- We found such a gauge for a magnetic unit cells with $q \times q+1$ electronic sites, $\chi(\boldsymbol{r}) = \sum_{i=1}^{N_v} s_i\phi(\boldsymbol{r},\boldsymbol{r}_i)$ with



$$\phi(z,z_i) = 2\text{Re}\left[\frac{(z-z_i)^2}{\tau_1\tau_2}\text{arctg}\left(\frac{i\tau_1}{\tau_2}\right)\right] + q\pi\text{Re}\left(\frac{z^2}{\tau_1\tau_2}\right)$$
$$-(q+1)\pi\frac{\text{Im}^2(z/\tau_2)\,\text{Re}(\tau_1/\tau_2)}{\text{Im}^2(\tau_1/\tau_2)} - q\pi\frac{\text{Im}^2(z/\tau_1)\,\text{Re}(\tau_2/\tau_1)}{\text{Im}^2(\tau_2/\tau_1)} \qquad (5)$$
$$+\pi\frac{\text{Im}(z/\tau_1)}{\text{Im}(\tau_2/\tau_1)} + \left[2\pi\text{Re}\left(\frac{z_i}{\tau_2}\right) - \pi\right]\frac{\text{Im}(z/\tau_2)}{\text{Im}(\tau_1/\tau_2)}$$

Almost anti-symmetric gauge

- The superconducting phase is accompanied by a vector potential that correspond to a homogeneous magnetic field and fulfill

$$\mathbf{J}(\boldsymbol{r}) = \mathbf{J}(\boldsymbol{r}+\boldsymbol{\tau}_i) \Rightarrow \mathbf{A}(\boldsymbol{r}+\boldsymbol{\tau}_i) = \mathbf{A}(\boldsymbol{r}) + \frac{1}{2}\boldsymbol{\nabla}\left[\Theta'(\boldsymbol{r}+\boldsymbol{\tau}_i) - \Theta'(\boldsymbol{r})\right] \qquad (6)$$

- We dub it the almost anti-symmetric gauge (AAG):

$$\mathbf{A} = \frac{2\Phi_0 N}{a_1 a_2\sin^2(\alpha_1-\alpha_2)}\left[\frac{(\boldsymbol{r}\times\hat{\boldsymbol{\tau}}_1)\times\hat{\boldsymbol{\tau}}_2}{q+1} + \frac{(\boldsymbol{r}\times\hat{\boldsymbol{\tau}}_2)\times\hat{\boldsymbol{\tau}}_1}{q}\right] \qquad (7)$$

- The AAG vs. the Landau gauge:

Vector potential	$\mathbf{A}(\boldsymbol{r}) = 2p\Phi_0\left(\frac{y}{q+1}, \frac{x}{q}\right)$	$\mathbf{A}(\boldsymbol{r}) = 2p\Phi_0\left(\frac{-y}{q+1}, 0\right)$
Total flux	$\Phi = \Phi_0 p, \quad 1 \le p \le q(q+1)$	$\Phi = q\Phi_0 p, \quad 1 \le p \le q+1$

Quasi-particle bands

- We use the following Bloch wave to obtain the electronic band structure:

$$\varphi_{\boldsymbol{k},s}(\boldsymbol{r}) = \frac{1}{\sqrt{N_1N_2}}\sum_{\boldsymbol{R}} e^{i\boldsymbol{k}\cdot\boldsymbol{R}}|\boldsymbol{R}+\boldsymbol{r},s\rangle$$

$$\boldsymbol{R} \equiv \boldsymbol{R}_{m_1,m_2} = m_1\boldsymbol{\tau}_1 + m_2\boldsymbol{\tau}_2$$

$$\boldsymbol{k} \equiv \boldsymbol{k}_{n_1,n_2} = \frac{2\pi n_1}{N_1|\boldsymbol{\tau}_1|}\hat{\boldsymbol{\tau}}_1 + \frac{2\pi n_2}{N_2|\boldsymbol{\tau}_2|}\hat{\boldsymbol{\tau}}_2$$

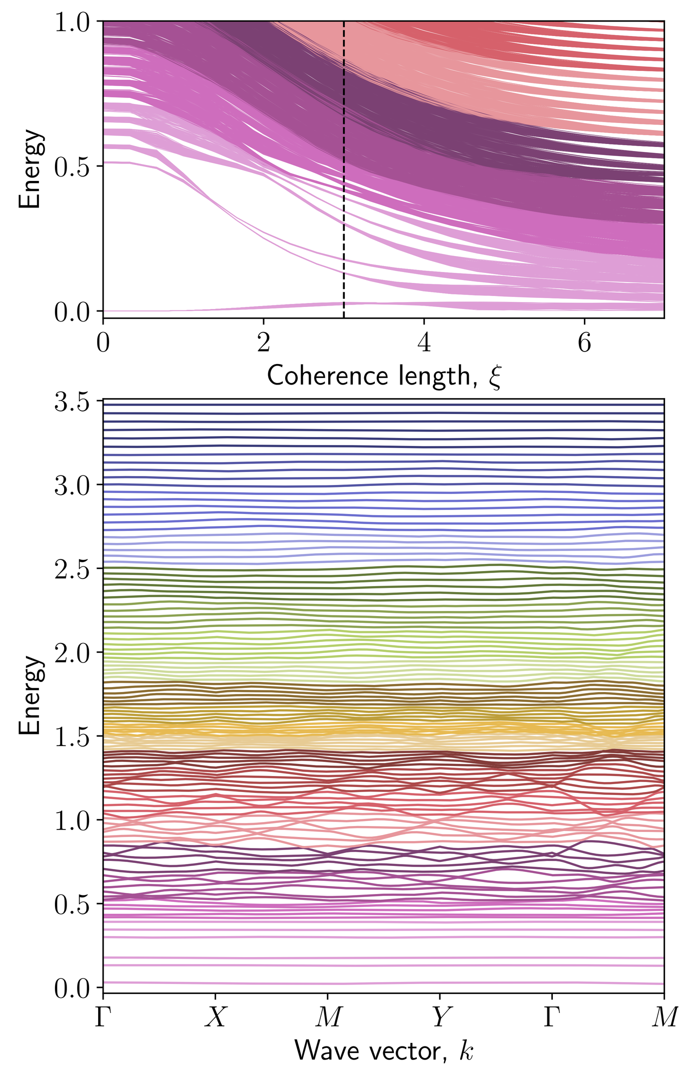


Figure 2: (Top) Quasi-particle bands as function of the coherence length, for a pinned vortex lattice in a p-wave superconductor. The magnetic unit cell contains 10×11 atomic sites and the vortices are placed along its diagonal with maximal separation. We take $t = |\Delta| = \mu = 1$. (Bottom) The electronic band structure of a p-wave superconductor with the coherence length set to $\xi = 2$.

Charge response

- c_{xy} is manifested in the effective action by a partial Chern-Simons term $S_{\text{pCS}} = \pm c_{xy}\int d\boldsymbol{r}dt\; a_t\left(\nabla\times\boldsymbol{a}\right)_z$ with $a_\mu = A_\mu - \partial_\mu\Theta/2$, $\mu \in \{t, x, y\}$.[2]
- Thus, in analogy to the Streda formula we have $c_{xy}(\boldsymbol{r}) = \pm\,\partial\rho(\boldsymbol{r})/\partial B_z|_{B_z=0}$ with $\rho(\boldsymbol{r}) = \delta S_{\text{eff}}/\delta a_t$ and $B_z = \left(\nabla\times\boldsymbol{a}\right)_z$. When taking the derivative in the Streda formula, we simultaneously flip the magnetic field and all the vorticities.

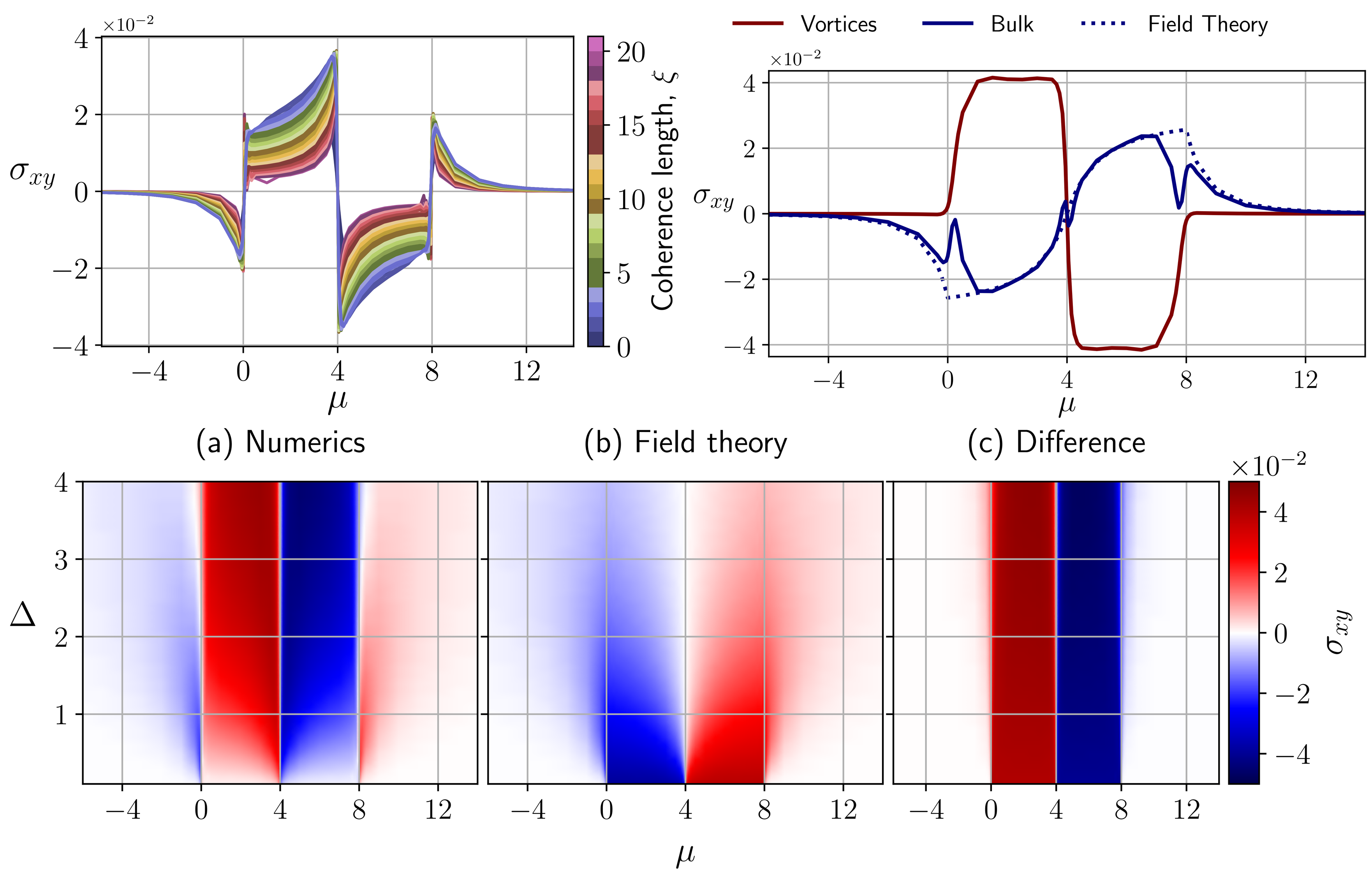


Figure 3: (Left) c_{xy} vs. μ for different ξ. (Right) We crudely separate the magnetic unit cell average of c_{xy} into a contribution from the vortices and a contribution from the bulk. (Bottom) c_{xy} vs. μ and Δ. The magnetic unit cell consists of 40×41 atomic sites, $t = |\Delta| = 1$ and $\xi = 2.5$. In addition, the magnetic unit cell contains 40×41 sites and two vortices that are pinned on its diagonal, partitioning it in a ratio of $1:2:1$.

References

[1] D. Ariad, Y. Avishai, and E. Grosfeld. How vortex bound states affect the hall conductivity of a chiral $p\pm ip$ superconductor. *arXiv:1603.00840*, 2018.

[2] D. Ariad, E. Grosfeld, and B. Seradjeh. Effective theory of vortices in two-dimensional spinless chiral p-wave superfluids. *Phys. Rev. B*, 92(3):035136, 2015.