Geometric Phases in Topological Superconductors

Thesis submitted in partial fulfillment
of the requirement for the degree of
"DOCTOR OF PHILOSOPHY"

by

Daniel Ariad

Submitted to the Senate of Ben-Gurion University
of the Negev

March 28, 2018

Beer-Sheva, Israel

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Approved by the Advisor

Approved by the Dean of the Kreitman School of Advanced Graduate Studies

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This work was carried out under the supervision of

Prof. Eytan Grosfeld

In the Department of Physics

Faculty of Natural Sciences

Research-Student's Affidavit when Submitting the Doctoral Thesis for Judgment

I, Daniel Ariad, whose signature appears below, hereby declare that

1. I have written this Thesis by myself, except for the help and guidance offered by my Thesis Advisor.
2. The scientific materials included in this Thesis are products of my own research, culled from the period during which I was a research student.

Date: November 5, 2018 Student's name: Daniel Ariad Signature: אריעד

Abstract

Toward the realization of a universal topological quantum computation (TQC) new methods have been developed to explore chiral superconductors in their topological phase, or topological superconductors. Vortex defects in 2D spinless chiral *p*-wave superfluid bind Majorana zero modes (MZMs) that endow these defects with non-Abelian exchange statistics. This property could be used to perform unitary transformations in the ground state manifold of the superconductor, which may then find applications in fault-tolerant quantum computation. Motivated by this potential, we developed an effective, low-energy theory for vortices in two-dimensional *p*-wave superfluids. In our derivation we used a single-valued gauge transformation that manifestly preserved the particle-hole symmetry of the action. Our theory reproduces the known physics of vortex dynamics such as the Magnus force proportional to the superconductor density. Moreover, the theory incorporates both complete and partial Chern-Simons terms. The former predicts a universal Abelian phase, $\exp(i\pi/8)$, associated with the exchange of two vortices. However, the phase has non-universal corrections attributed to the partial Chern-Simon term that are screened in charged superfluids.

Several types of systems exist in which the exchange of MZMs may be practically implemented. Arguably, the most relevant system is the topological Josephson junction as it is a relatively straightforward matter to experimentally control the motion of Josephson vortices. It has been previously suggested that Josephson vortices in topological Josephson junctions (TJJ) constitute such MZMs and retain the exchange statistics of bulk vortices. In this thesis we propose an effective Hamiltonian describing the collective motion of a phase soliton within a Josephson junction separating two topological superconductors. Then we derive the equations of motion for the soliton trapped in an annular Josephson junction and calculate the universal phase accumulated as it encircles the junction. We find that the universal phase depends on the parity of the number of vortices enclosed by the junction, then demonstrate that the presence of this phase can be measured through its effect on the junction's voltage characteristics.

A necessary step toward the long-term goal of storing and manipulating quantum information in a Hilbert space spanned by MZMs bound to vortex cores is to understand and characterize the vortices' electronic properties. With this objective in mind, we present a framework that incorporates (a) a general construction for the phase of a complex order parameter capable of encoding any configuration of vortex defects residing on a flat torus or cylinder and (b) a gauge for the vector potential, dubbed "the almost anti-symmetric gauge," that allows, in a system with periodic boundary conditions, access to the highest resolution for its magnetic field dependence. We use this framework together with Bloch's theorem to solve a tight-binding Bogoliubov-de Gennes Hamiltonian for an infinite two-dimensional vortex lattice in a chiral p-wave superconductor. This, in turn, allows us to access the dispersion of quasi-particle states and study the formation of Caroli-de Gennes-Matricon states and sub-gap bands induced by tunneling between vortices. In addition, we generalize the Streda formula to account for the charge response, c_{xy} of a chiral p-wave superconductor. We show that c_{xy} is a sum of two contributions, one which is non-universal and the other equals $\kappa/8\pi$, where κ is the Chern number of the superconductor.

Keywords: Topological superconductors, Topological superfluids, Annular Josephson junction, Effective low energy theory, Chern-Simons term, Vortex defects, Abelian exchange phase, Topological spin

Contents

Chapter 1

Introduction

1.1 Overview

Chiral superconductors constitute a class of unconventional superconductors in which Cooper pairs spontaneously develop an angular momentum. They superconductors attract attention in part because, in their topological phase, they may host Majorana zero modes (MZMs). These MZMs exhibit a number of intriguing physical properties that can be exploited to encode and manipulate quantum information in such a way that is robust to decoherence[1]. Another of their unique properties is the existence of surface currents, carried both by edge modes and by bulk states near the surface[2]. Although the edge currents are not quantized, the edge states can give rise to a quantized thermal Hall conductance[3, 4, 5]. Another striking phenomena that can occur in triplet chiral superconductors is the nucleation of half-quantum vortices that carry half the superconducting magnetic flux quantum[4, 6].

The purpose of this research is to explore the fundamental properties of vortices in topological superconductors, calculate geometric phases that accompany a adiabatic exchange of vortices, suggest experiments by which to measure these phases, and formulate using Pfaffian algebra the Berry theorem for paired states.

The distinctive features of chiral superconductivity may be found in the p-wave superconductor in a spinless one-band setting. Therefore we begin by deriving an effective action of this model that accounts for vortices, the same vortices that have recently attracted considerable interest for their trapping of MZMs. It is only when superconducting vortices bind MZMs that, by virtue of the properties of the superconductor, the composite object of the vortex and MZM satisfies non-Abelian exchange statistics and can be employed as building blocks for TQC. The field theory we found for this chiral p-wave superconductor incorporates a partial Chern-Simons (CS) term, as well as a complete CS term. Remarkably, the coefficient of the complete CS term is proportional to the Chern number. While we did discover that the field associated with the complete CS term is related to the nucleation of vortices, the physical meaning of its corresponding particle density remains a puzzle.

Quantum computation based solely on MZMs is not universal unless it is supplemented by a $\pi/8$-phase gate. Using the Abelian universal exchange phase for vortices, it has been argued that such a gate can be generated[7, 8]. To this end, achieving this goal, we focused our attention on phase solitons in annular topological Josephson junctions[9], deriving the action governing their dynamics and extracting from the equations of motion of the action a universal quantum exchange phase for the solitons. The phase manifests itself as a specific spectral feature, therefore carries experimental significance. Moreover, we showed that one can induce a persistent motion of these solitons, in turn resulting in a measurable voltage signal, by trapping them in an annular Josephson junction and adding a vortex within the loop. This spectral feature is in sharp contrast to that of solitons in non-topological annular Josephson junctions, which are not affected by the presence of bulk vortices in the inner superconductor.

Finally, we investigated the structure and electronic properties of the vortex-bound states in a spinless p-wave superconductor by using the tight-binding model. We used the Streda formula to calculate the anomalous charge response, c_{xy}, at the bulk, and verified that it agrees with our field theory predictions. Surprisingly, we found that the contribution to c_{xy} from the vortices is quantized. We attribute this contribution to the formation of bound states as well as deduce from it the quantum phase of the physical system. This effect is unexpected apriorily since, according to the field theory, the electromagnetic vector field is decoupled from the vortices. Our investigation reveals that details of the vortex cores are required in order to account for this effect, details that are absent in the field theoretical formulation, in which vortices are treated as point-like objects.

1.2 Scientific Background

The TQC scheme relies on adiabatic braiding of non-Abelian anyons to generate quantum computation. Among non-Abelian anyon models, Majorana fermions are arguably the closest to realization. The 2D , spinless chiral p-wave superconductor is the simplest model that can be used to describe a topological superconductor. Vortices in this type of superconductor are expected to host zero-energy, localized Majorana fermions and therefore play significant role in proposals for universal TQC[10, 1].

The earliest theoretical investigations of p-wave superconductivity focused on 'intrinsic' p-wave superconductors, such as Strontium Ruthenate (Sr_2RuO_4) and the $\nu = 5/2$ quantum Hall effect (QHE), that is mappable to a p-wave superconductor by a CS transformation. The Sr_2RuO_4 is a highly anisotropic, layered material with three bands crossing the Fermi energy[11, 12]. In addition, it has a weak to intermediate spin-orbit coupling that does not break the spin degeneracy of topological surface states. Hence the Bogoliubov quasiparticles on its surface are spin-degenerate, thus half-quantum vortices are required in order to nucleate isolated MZMs. The existence of such vortices has not yet been established[13]. In the $\nu = 5/2$ QHE, the controversy surrounding the nature of its state and complexity of its effective description tends to mask clean signatures of MZMs[14, 15], though much progress has been made in determining its correct ground state[16, 17]. One possibility of overcoming such issues involves the fabrication of heterostructures in which an interface between a topological insulator and an s-wave superconductor can be mapped into a spinless p-wave superconductor[18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. Many proposals involving the use of other materials have been made, showing that a topological insulator can essentially be replaced by a 2D quantum well with either Zeeman or Dresselhaus coupling and Rashba spin-orbit coupling[28, 29, 30, 31, 32, 33, 34]. Another promising route to topological superconductivity is to deposit magnetic atoms on the surface of an s-wave superconductor with a strong spin-orbit coupling[35, 36, 37].

MZMs have been suggested as the basis for topological quantum computing, with computational steps made by physically exchanging (braiding) positions of the quasiparticles, as is illustrated in Figure 1.1. In principle, the resultant state depends only on the topology of the exchange; the physical system is therefore argued to be robust against local perturbations. In a 2D p-wave superconductor vortex defects bind MZMs that endow them with non-Abelian exchange statistics. The braiding of MZMs may result in a unitary transformation in the space of ground-state degenerate manifold. However, braiding operations we can only perform single-qubit rotations by an angle $\pi/2$ in the Hilbert space, which are not rich enough to support all the gates required for a universal quantum computer[38, 39, 40]. The Abelian statistics, attributed to the above-mentioned vortices, can supplement the missing operation by generating the $\pi/8$-phase gate, completing a universal gate set. This gate can be generated in a topologically protected manner by performing certain operations that change the topology of the system [41, 42, 7]. It is therefore quite important to formulate a cogent theory that accounts for the dynamics of vortices in p-wave superconductors.

In order to shed light on the collective response of the 2D spinless chiral p-wave superconductor to external electromagnetic fields, a low-energy effective action has been derived by the standard gradient expansion method [43, 44, 45, 2, 46, 47]. However, in this derivation vortices have generally been left out. It appears then that the Abelian exchange phase of vortices, while surmised from the conformal properties of its edge states or the properties of candidate bulk wavefunctions [48, 4, 49], has never been derived from a microscopic model [50, 51]. In particular, it has generally been accepted that its value is universal. *We showed that, in standard derivations of the action of p-wave superconductors, a crucial term that is directly responsible for this universal exchange phase is, in fact, lacking[52].*

Realization of itinerant, non-abelian quasi-particles is the much-coveted goal of a large community of physicists exploring topological states of matter[53]. It has been suggested that Josephson vortices in topological Josephson junctions (TJJ) would constitute such MZMs and retain the universal exchange statistics of bulk vortices[9]. In contrast to many other systems, it is a relatively straightforward matter to allow for braiding by experimentally controlling the motion of Josephson vortices[54, 55, 56]. This braiding process could lead to spectral signatures of non-Abelian ex-

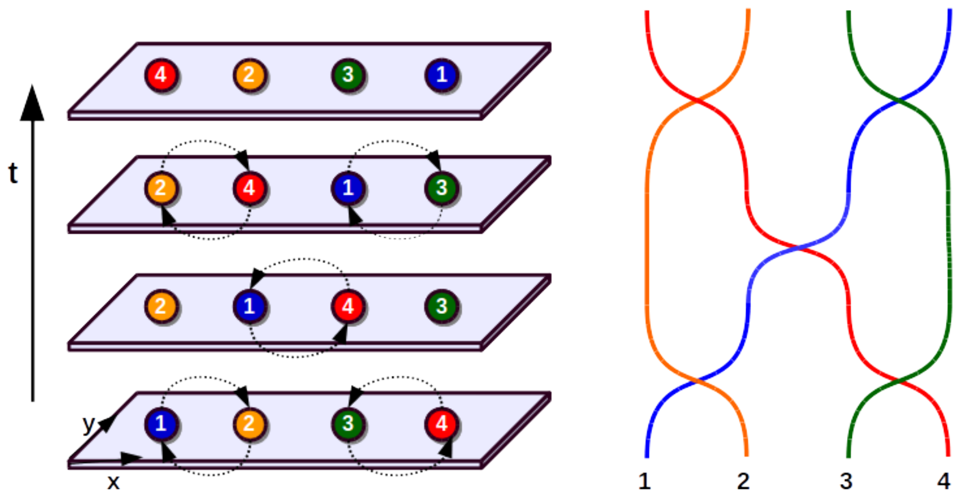

Figure 1.1: **Braiding Majorana Fermions.** The spatial and space-time trajectories of a series of exchanges resulting complicated braid of worldlines.

change statistics. Experimental and theoretical investigations of TJJs are now in progress[57, 58]. *Although the TJJs are a promising way of exchanging MZMs, our study reinforces the hypothesis that Josephson vortices retain the exchange statistics of bulk vortices.*[59].

Many aspects of vortex-Majorana lattices in a 2D chiral p-wave superconductor have been investigated extensively [60, 61, 62, 63, 64, 65]. In addition, using the tight-binding model, the structure of vortex-bound states in a spinless p-wave superconductor was studied[66, 67, 68, 69]. It was found that the vortex Caroli-de Gennes-Matricon (CdGM) bound states play an important role in the accumulation of charge in the vortex core [70] as well as that vortices and anti-vortices accumulate different charges[71, 72]. We recently showed that the anomalous charge response, c_{xy} is a sum of two contributions, one which is non-universal and the other equals $\frac{\kappa}{8\pi}$, where κ is the Chern number of the superconductor. Moreover, we note that c_{xy} is proportional to the anomalous Hall conductivity, which in turn is proportional to the polar Kerr angle. *Thus, these results should affect calculation of the polar Kerr effect, hence they are significant for the determination of the order parameter of superconductors.* [47, 73, 74, 75, 76, 77].

In the rest of this chapter we will introduce several key topics, including the exchange statistics of Majorana bound states, a path-integral description of a chiral p-wave superconductor, and the topological annular Josephson junction in the presence of a soliton. In addition, we will present the Thouless representation of the p-wave many-body groundstate and give a formula for the Berry phase (Abelian case) in terms of this groundstate.

1.2.1 The exchange statistics of Majorana bound states

Finding materials that admit a topological phase is the first step in creating quantum devices that are topologically protected, as any material must accommodate qubits for encoding quantum information and quantum gates to manipulate them. Quantum gates may be implemented by adiabatic, or the slow braiding, of topological defect worldlines, that end with their return to their original positions. The braiding process transforms one ground-state to another state in the same degenerate subspace. All braids can be built up from a finite set of elementary exchanges, thus exchange statistics can be specified by the unitary matrices representing the effect of these elementary exchanges on the groundstate manifold.

Up to a global phase, the non-Abelian statistics of MZMs can be inferred from the action of braid group generators on MZM operators [38]. The action of the braid group generator T_i on MZM operators is

$$T_i : \begin{cases} \gamma_i \longrightarrow \gamma_{i+1}, & ; \\ \gamma_{i+1} \longrightarrow -\gamma_i, & ; \\ \gamma_j \longrightarrow \gamma_j, & \text{for } j \neq i \text{ and } j \neq i+1. \end{cases} \tag{1.1}$$

By solving the BdG equation, these topological properties of Majorana operators were shown to exist in p-wave superconductor ground states[78]. In its ground state each vortex defect hosts one MZM which can be combined into n complex Dirac fermions, thereby giving rise to the degeneracy of the ground state equal to 2^{n-1} for a fixed parity of particle number (i.e., each fermionic level may be either filled or empty)[4, 38].

1.2.2 Path-Integral Description of a chiral p-wave superconductor

In a p-wave superconductor, those quasi-particles that exhibit non-Abelian statistics are flux $h/2e$ vortices [79, 80]. We would like to be able to “integrate out” the Fermionic degrees of freedom and make a loop expansion around a ”bare” Green’s function to obtain a low-energy theory [81, 82]. In the following we introduce the basics of path-integral theory.

The Bogoliubov–de Gennes Hamiltonian for a spinless p-wave superconductor is [43, 2]

$$\mathcal{H} = \int_{\mathbb{R}} dt \int_{\mathbb{R}^2} d\mathbf{x}\, \psi^{\dagger}_{\mathbf{x},t} \left(\frac{(\mathbf{p}-\mathbf{A})^2}{2m} - \mu - A_0 \right) \psi_{\mathbf{x},t} + \frac{1}{2} \left[\psi_{\mathbf{x},t} \left\{ \bar{\Delta}, \mathbf{p_x} + i\mathbf{p_y} \right\} \psi_{\mathbf{x},t} + \text{h.c.} \right], \tag{1.2}$$

where $\mathbf{p} = -i\boldsymbol{\nabla}$ is the momentum, ψ, $\psi^{\dagger}$ are the electron field operators, Δ is the order parameter that depends on space and time ,and $\mathbf{A}(\mathbf{r})$ is the electromagnetic vector potential. From here on we assume $e = \hbar = 1$. The action functional corresponding to the Hamiltonian Eq. (1.2) is

$$\mathcal{S}\left(\bar{\phi}_{\mathbf{x},t}, \phi_{\mathbf{x},t}\right) = \int_{-\infty}^{\infty} dt \int_{\mathbf{R}^2} d\mathbf{x} \left[\bar{\phi}_{\mathbf{x},t} \left(i\partial_t\right) \phi_{\mathbf{x},t} - \mathcal{H}\left(\bar{\phi}_{\mathbf{x},t}, \phi_{\mathbf{x},t}\right) \right], \tag{1.3}$$

where the fermion operators appearing in Eq. (1.2), $\psi^{\dagger}_{\mathbf{x},t}$ and $\psi_{\mathbf{x},t}$, were replaced by Grassmann fields, denoted by $\bar{\phi}_{\mathbf{x},t}$ and $\phi_{\mathbf{x},t}$, respectively. The partition function of the system is given by the sum over all possible Grassmann field configurations, weighted by the action functional of the fields,

$$\mathcal{Z} = \int \mathcal{D}(\bar{\eta}_{\mathbf{x},t}, \eta_{\mathbf{x},t}) e^{i\mathcal{S}(\bar{\eta}_{\mathbf{x},t}, \eta_{\mathbf{x},t})}. \tag{1.4}$$

The action is quadratic in the Grassmann fields and the partition function can be straightforwardly integrated out. We use Nambu notation

$$\eta_{\mathbf{x},t} = \begin{pmatrix} \phi_{\mathbf{x},t} \\ \bar{\phi}_{\mathbf{x},t} \end{pmatrix} \text{ and } \bar{\eta}_{\mathbf{x},t} = \begin{pmatrix} \bar{\phi}_{\mathbf{x},t}, \phi_{\mathbf{x},t} \end{pmatrix}. \tag{1.5}$$

Writing the action in terms of Nambu spinors gives

$$\mathcal{S}\left(\bar{\eta}_{\mathbf{x},t}, \eta_{\mathbf{x},t}\right) = \frac{1}{2} \int_{-\infty}^{\infty} dt \int_{\mathbf{R}^2} d\mathbf{x} \left[\bar{\eta}_{\mathbf{x},t} \mathcal{G}^{-1} \eta_{\mathbf{x},t} \right], \text{ where } \mathcal{G}^{-1} = i\partial_t - \mathcal{H} \tag{1.6}$$

In terms of the Pauli matrices, The inverse Green matrix in the presence of electromagnetic fields is

$$\mathcal{G}^{-1} = i\partial_t - \tau_3 \left(\frac{(\mathbf{p} - \tau_3 \mathbf{A})^2}{2m} - \mu - A_0 \right) - \frac{1}{2}\tau_1 \{\Delta, \mathbf{p}_x\} - \frac{1}{2}\tau_2 \{\Delta, \mathbf{p}_y\} \tag{1.7}$$

where $\boldsymbol{\tau}_i$ are the Pauli matrices and the order parameter is $\Delta = \Delta_0 e^{i\tau_3\theta(\mathbf{x},t)}$.

The functional integration over a Gaussian of real Grassmann fields is

$$\mathcal{Z} = \int \mathcal{D}(\bar{\eta}_{\mathbf{x},t}, \eta_{\mathbf{x},t}) e^{i\mathcal{S}(\bar{\eta}_{\mathbf{x},t},\eta_{\mathbf{x},t})} = \prod_{\mathbf{x},t} \mathrm{Pf}\left(\mathcal{G}^{-1}_{\mathbf{x},t}\right) = \exp\left[\frac{1}{2}\mathrm{Tr}\log\left(\mathcal{G}^{-1}_{\mathbf{x},t}\right)\right], \tag{1.8}$$

where TrA stands for $\sum_{\mathbf{x},t}\langle \mathbf{x}, t|\mathrm{tr}A|\mathbf{x}, t\rangle$ and tr is the trace over the 2×2 Nambu space [83].

1.2.3 The topological annular Josephson junction in the presence of a soliton

In order to derive the effective Hamiltonian of the topological annular Josephson junction, we considered the Josephson junctions described in Fig.(1.2). Josephson vortices are trapped in insulating regions between superconductors and are solutions of the sine-Gordon equation; thus its the order parameter is complex and its phase, in the weak coupling limit, obeys the sine-Gordon equation. In the case of topological superconductors, such vortices can bind a localized MZM; this despite the fact that they lack a normal core. We then linearize the corresponding Hamiltonian for each of the edges and add a coupling term which allows for tunneling of Majoranas; the Majorana tunneling term is found by taking the overlap between the two edge states. Only in the case of counter-propagating Majorana edge states a localized MZM would appear in the Josephson vortex[9].

The Hamiltonian of the topological annular Josephson junction with a moving soliton is

$$\mathcal{H} = \int \mathrm{d}x \Psi^\dagger_x H \Psi_x, \tag{1.9}$$

where $\Psi_x = (\psi_x, \bar{\psi}_x)^T$ is a spinor which consists of a periodic and an anti-periodic Majorana field (i.e., the fields are self-adjoint), respectively. The single particle Hamiltonian is

$$H = \tau_z i v \partial_x - \tau_y W(x, q), \tag{1.10}$$

with $W(x) = m(q)\cos[\pi(x - q)/L]$ being the order parameter for a short Josephson vortex.

The representation of the Majorana field depends on its boundary conditions as follows

$$\psi_x = \frac{1}{\sqrt{L}} \sum_{k_p} e^{-ik_p x}\psi_{k_p}, \quad \bar{\psi}_x = \frac{1}{\sqrt{L}} \sum_{k_a} e^{ik_a x}\bar{\psi}_{k_a}, \tag{1.11}$$

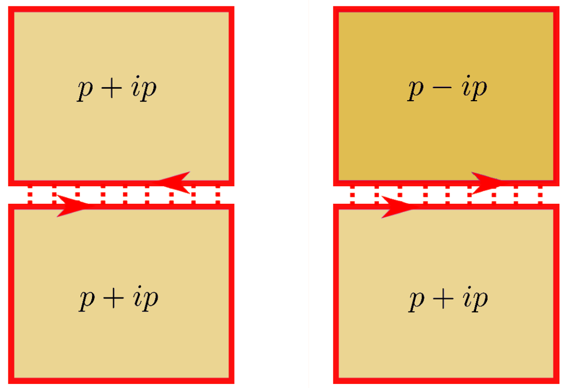

Figure 1.2: **Josephson junctions.** Left panel: $p_x + ip_y$ / $p_x + ip_y$ junction. Right panel: $p_x - ip_y$ / $p_x + ip_y$. The direction of the propagation of Majorana edge states is marked by arrows. Dotted lines indicate electron tunneling.

where

$$k_p(m) = \frac{2\pi}{L} m, \quad k_a(n) = \frac{2\pi}{L}\left(n + \frac{1}{2}\right), \quad m, n \in \mathcal{Z}. \tag{1.12}$$

The opposite signs of these exponents reflect counter-propagating Majorana edge states. For numerical purposes we set a cutoff $k_p(n_{min}) \leq k_p \leq k_p(n_{max})$, $k_a(n_{min}) \leq k_a \leq k_a(n_{max} - 1)$.

1.2.4 Thouless representation of the Hartree-Fock-Bogoliubov groundstate

The Hartree-Fock-Bogoliubov (HFB) groundstate can be represented as

$$|\Omega\rangle = A \exp\left(\sum_{i<j} Z_{ij} \psi_i^\dagger \psi_j^\dagger\right) |0\rangle \tag{1.13}$$

where ψ_i is a fermion annihilation operator satisfying $\psi_i |0\rangle = 0$, $Z = (VU^{-1})^*$ is a skew-symmetric matrix, and the columns of the block matrix $(U\ V)^T$ are eigenstates that correspond to positive eigenenergies in ascending order. Moreover, $A = \sqrt{|\det U|}$ is a normalization constant, ensuring that $\langle\Omega|\Omega\rangle = 1$. This representation is known in the literature as the Thouless Representation[84, 85].

In the construction of the HFB groundstate we assume that all positive energy single-particle eigenstates are related to the negative ones by virtue of particle-hole symmetry, $|\epsilon\rangle = \tau_x K|-\epsilon\rangle$, where K is complex a conjugate operator and τ_x is the first Pauli matrix in Nambu space. In the absence of degenerate eigenstates, this relation is automatically fulfilled. However, subspaces of degenerate eigenstates should be spanned by states that fulfill the relation. For example, in the case of two degenerate zero-energy states, we need to construct two orthonormal states, $\langle v_i|v_j\rangle = \delta_{i,j}$, that are also particle-hole counterparts of one other, $|v_2\rangle = \tau_x K|v_1\rangle$.

The case of zero-modes is more complicated than that of degeneracies at higher-energies because it is not known a priori which one of the zero-modes should participate in the many-body groundstate. In order to choose the correct zero-mode we check that groundstate is not orthogonal to the bare vacuum, $\langle 0|\Omega\rangle = \sqrt{|\det U|} \neq 0$. Otherwise U would be singular, resulting in an ill-defined Thouless representation.

The overlap between two HFB groundstates is given by

$$\langle\Omega_1|\Omega_2\rangle = A_1 A_2 S_N \mathrm{pf}\,\mathcal{Z} \tag{1.14}$$

where A_i represent the normalization constants, $S_N = (-1)^{N(N+1)/2}$ and

$$\mathcal{Z} = \begin{pmatrix} Z_2 & -\mathbb{I} \\ \mathbb{I} & -(Z_1)^* \end{pmatrix} \tag{1.15}$$

is a $2N \times 2N$ skew-symmetric matrix with $Z_i = \left(V_i U_i^{-1}\right)^*$ and $i = 1, 2$.

1.2.5 The Berry phase (Abelian case)

We are primary interested in the geometric phase, or the Berry phase, accompanying an adiabatic exchange of vortices over time t. We parametrize the process using a set $\mathbf{R}$ so that the Berry phase acquires the form

$$\gamma_n = i\int_{t_i}^{t_f} \mathrm{d}t\ \langle n(\mathbf{R}(t'))|\nabla_{\mathbf{R}}|n(\mathbf{R}(t'))\rangle\dot{\mathbf{R}} = i\oint_C d\mathbf{R}\langle n(\mathbf{R})|\nabla_{\mathbf{R}}|n(\mathbf{R})\rangle. \tag{1.16}$$

The time independent wave function, $|n(\mathbf{R})\rangle$, is defined uniquely up to a global phase which can be gauged. Under the gauge transformation $|n(\mathbf{R})\rangle \to e^{i\zeta(\mathbf{R})}|n(\mathbf{R})\rangle$, where $\zeta(\mathbf{R})$ must maintain the smoothness and the single-valueness of the wave function. Under the same transformation the Berry vector potential $\mathbf{A}_n(\mathbf{R})$ is transformed as $A_n(\mathbf{R}) \to A_n(\mathbf{R}) - \boldsymbol{\nabla}_{\mathbf{R}}\zeta(\mathbf{R})$. Consequently, the Berry phase will change by $\Delta\gamma_n = -\oint_C \mathrm{d}\mathbf{R}\boldsymbol{\nabla}_{\mathbf{R}}\zeta(\mathbf{R}) = \zeta(\mathbf{R}(t_i)) - \zeta(\mathbf{R}(t_f)) = 2\pi m$ with m being an integer. This last equality is a result of $\mathbf{R}(t_i)$ and $\mathbf{R}(t_f)$, referring to the same point in the parameter space while $\zeta(\mathbf{R})$ is allowed to be multivalued so long as the wave function is kept single-valued.

The Berry phase of a HFB groundstate along a closed path is $\gamma = \int_C d\mathbf{R} \cdot \mathcal{A}(\mathbf{R})$ with

$$\mathcal{A}(\mathbf{R}) = i\langle\Omega|\nabla_{\mathbf{R}}\Omega\rangle = \frac{i}{4}\mathrm{tr}\left((1 + Z^\dagger Z)^{-1}(Z^\dagger Z' - Z'^\dagger Z)\right), \tag{1.17}$$

$Z = (VU^{-1})^*$ and the columns of the block matrix $(U\ V)^T$ are single-particle eigenstates corresponding to positive eigenenergies in ascending order[86]. For numerical purposes it is preferable to write the Berry connection without derivatives of inverse matrices,

$$\mathcal{A}(\mathbf{R}) = i\langle\Phi|\nabla_{\mathbf{R}}\Phi\rangle = \frac{i}{4}\mathrm{tr}\left[V^\dagger V' - V\left(V^\dagger\right)' + \left(U^\dagger\right)^{-1} V^\dagger V\left(U^\dagger\right)' - V^\dagger V U^{-1} U'\right]^*. \tag{1.18}$$

The complete derivation is in Appendix D.

1.3 Short summary of the papers

(a) On the effective theory of vortices in two-dimensional spinless chiral p-wave superfluids

We calculated the Abelian exchange phase associated with the adiabatic exchange of vortices using a microscopic model of a *p*-wave superconductor, determining that, in the standard derivations of the action of p-wave superconductors, a crucial term that is directly responsible for this universal exchange phase is lacking. By carefully sorting out the components of the theory's gauge structure, we showed how to produce the missing term of the CS type in the action. This allowed us to predict the conditions under which the exchange phase will deviate from its universal value. We believe that this new understanding will play an important role in harnessing Majorana fermions toward universal quantum computation.

Ariad, Daniel, Eytan Grosfeld, and Babak Seradjeh. "Effective theory of vortices in two-dimensional spinless chiral p-wave superfluids." Physical Review B 92.3 (2015): 035136.

(b) Signatures of the topological spin of Josephson vortices in topological superconductors

Realization of non-abelian quasi-particles, known as Majorana fermions, is an ongoing challenge for physicists exploring topological states of matter. Toward achieving this goal, we recently suggested that Josephson vortices in TJJ would constitute such MZMs and retain the exchange statistics of bulk vortices. In order to corroborate this hypothesis, we found the universal exchange phase of Josephson vortices by developing a procedure to calculate the Berry connection of systems possessing particle-hole symmetry. This calculation confirmed that the Abelian phase resulting from the exchange between a bulk vortex and a Josephson vortex is $\pi/8$. In addition, we suggested an experiment by which to measure the presence of this phase.

Ariad, Daniel, and Eytan Grosfeld. "Signatures of the topological spin of Josephson vortices in topological superconductors." Physical Review B 95.16 (2017): 161401.

(c) How vortex bound states affect the Hall conductivity of a chiral $p \pm ip$ superconductor

This work extend our understanding of the anomalous charge response, c_{xy} of chiral superconductors. It is established that in order to correctly apply the Streda formula for calculating c_{xy} it is necessary to employ compact geometries that avoid edge effects. This, in turn, requires a careful analysis of the effect of finite-radius vortex nucleation that leads to an adjustment of the Streda formula. The modified Streda formula is then applied to calculate c_{xy} for a $p_x \pm ip_y$ superconductor placed on a square lattice at zero magnetic field and zero vorticity. We show that c_{xy} is a sum of two contributions, one which is non-universal and the other equals $\kappa/8\pi$, where κ is the Chern number of the superconductor. Moreover, we note that c_{xy} is proportional to the anomalous Hall conductivity, which in turn is proportional to the polar Kerr angle. Thus, these results should affect the calculation of the polar Kerr effect, hence they are significant for the determination of the order parameter of superconductors.

Chapter 2

Publications

2.1 Effective theory of vortices in two-dimensional spinless chiral p-wave superfluids

Daniel Ariad, Eytan Grosfeld, and Babak Seradjeh

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Effective theory of vortices in two-dimensional spinless chiral p-wave superfluids

Daniel Ariad and Eytan Grosfeld*
Department of Physics, Ben-Gurion University of the Negev, Beer-Sheva 8410501, Israel

Babak Seradjeh
Department of Physics, Indiana University, Bloomington, Indiana 47405, USA
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We propose a $\mathbb{U}(1) \times \mathbb{Z}_2$ effective gauge theory for vortices in a $p_x + ip_y$ superfluid in two dimensions. The combined gauge transformation binds $\mathbb{U}(1)$ and $\mathbb{Z}_2$ defects so that the total transformation remains single-valued and manifestly preserves the particle-hole symmetry of the action. The $\mathbb{Z}_2$ gauge field introduces a complete Chern-Simons term in addition to a partial one associated with the $\mathbb{U}(1)$ gauge field. The theory reproduces the known physics of vortex dynamics such as a Magnus force proportional to the superfluid density. More importantly, it predicts a universal Abelian phase, $\exp(i\pi/8)$, upon the exchange of two vortices. This phase is modified by nonuniversal corrections due to the partial Chern-Simon term, which are nevertheless screened in a charged superfluid at distances that are larger than the penetration depth.

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I. INTRODUCTION

The two-dimensional spinless chiral p-wave superfluid is the minimal model for describing the properties of many realizations of topological superfluids and superconductors: topological insulator-superconductor interfaces [1–3], the layered material Sr_2RuO_4 [4–6], some cold atom systems [7,8], and certain spin models admitting anyon excitations [9]. In this model, the vortex defects of the phase of the pairing order parameter bind Majorana zero modes that endow them with non-Abelian exchange statistics [10–13]. Thus they have been proposed as potential candidates for fault-tolerant, topological quantum information processing [14–16]. In addition, they are expected to admit a quantized Abelian exchange phase that plays an important role in proposals for universal topological quantum computation with vortices [17]. It is therefore quite important to formulate a cogent theory that accounts for the dynamics of vortices.

In previous work on this system, a low-energy effective action has been derived by the standard gradient expansion method [18–23], shedding light on the collective response of the superfluid to external electromagnetic fields. However, in this derivation vortices have been generally left out. It appears then that the Abelian exchange phase of vortices, while surmised from the conformal properties of its edge states or the properties of candidate bulk wave functions [10,12,24], has never been derived from a microscopic model [25,26]. Consequently, it remains unclear whether bulk vortices in a chiral p-wave superfluid or superconductor exhibit this exchange phase and, if so, to what degree it is universal or how it is affected by the physics of the system.

To answer these questions, in this paper, we derive a $\mathbb{U}(1) \times \mathbb{Z}_2$ effective gauge theory that handles vortex defects properly. The $\mathbb{U}(1)$ gauge field is governed by an action that is identical to the one previously derived by gradient expansion, including a partial Chern-Simons (CS) term. Interestingly, a $\mathbb{Z}_2$ gauge field emerges in the effective theory governed by a new *full* Abelian CS term. We show that the coefficient of the partial CS term is not a universal quantity and depends on the details of dispersion and higher-energy behavior of the system. The full CS term of the $\mathbb{Z}_2$ gauge field is, on the other hand, a truly topological term with a quantized coefficient. We calculate the exchange angle of two vortices due to each CS term and show that the new CS term dictates a universal Abelian exchange statistics phase of the vortices equal to $e^{i\pi/8}$. In contrast, for neutral superfluids, the partial CS term spoils the quantization of the exchange phase by adding a long-distance nonuniversal correction. For charged superfluids, screening effects exponentially diminish the latter over the effective penetration depth. This sets a low bound for the distance between vortices during exchange processes required for topological quantum computation.

II. GAUGE TRANSFORMATION

We start with the action for a spinless chiral p-wave superconductor [27], $Z = \int \mathcal{D}(\bar{\eta},\eta)e^{iS}$, where $\eta = (\phi,\bar{\phi})^{\mathsf{T}}$ and $\bar{\eta} = (\bar{\phi},\phi)$ are the Nambu spinors with Grassmann variables $\phi(r)$ and $\bar{\phi}(r)$ in the coordinate space $r = (\mathbf{r},t)$. In the following, we will interchangeably use $z \equiv t$ as the third coordinate and $d^3r = d\mathbf{r}dt$. The action is $S = \frac{1}{2}\int d^3r\ \bar{\eta}\mathcal{G}^{-1}\eta$, with $\mathcal{G}^{-1} = i\partial_t - \mathcal{H}$ the inverse Green's function matrix and the Bogoliubov–de Gennes Hamiltonian density [28],

$$\mathcal{H} = \begin{pmatrix} \xi_{\mathbf{p}-\mathbf{A}} - A_t & e^{i\theta/2}\Delta(\mathbf{p})e^{i\theta/2} \\ e^{-i\theta/2}\Delta(\mathbf{p})^\dagger e^{-i\theta/2} & -\xi_{\mathbf{p}+\mathbf{A}} + A_t \end{pmatrix}. \quad (1)$$

Here, $\xi_{\mathbf{p}}$ is the dispersion of excitations above the ground state, $\mathbf{p} = -i\nabla$ is the momentum operator, $\Delta(\mathbf{p})$ is the amplitude and $e^{i\theta(\mathbf{r},t)}$ is the phase of the superconducting order parameter (including vortices), and $A = (\mathbf{A},A_t)$ is the electromagnetic gauge field. (In a neutral superfluid, $A = 0$.) We assume $e = c = \hbar = 1$. In the continuum, $\xi_{\mathbf{p}} = \mathbf{p}^2/2m - \epsilon_F$ with ϵ_F the Fermi energy and $\Delta(\mathbf{p}) = v(p_x + ip_y)$ with v the slope of the pairing order parameter in momentum space.

In order to keep track of the winding number around each vortex we define $\theta(\mathbf{r},t) = \sum_{j=1}^{n}\theta_j(\mathbf{r},t)$, where

*grosfeld@bgu.ac.il

$\theta_j = \arg(\mathbf{r}-\mathbf{x}_j) \in (2\pi\ell_j, 2\pi(\ell_j+1)]$ is the phase around the vortex located at $\mathbf{x}_j(t)$ and ℓ_j is its winding number. We take the branch cut of $\arg(\mathbf{r})$ to be the positive real axis and index the corresponding Riemann sheets with the branch number $\ell \in \mathbb{Z}$ [29].

The partition function is invariant under a unitary transformation, U, of the inverse Green's function with a Jacobian of unit modulus; that is, $U = e^{i\alpha_0}e^{i\alpha^\mu\tau_\mu}$, where τ_x, τ_y, and τ_z are Pauli matrices in the Nambu space. We demand that U respect the particle-hole symmetric structure of the spinor fields. This means that U must transform $(\bar\eta,\eta)$ in such a way that ensures one spinor remains the conjugate transpose of the other and each component of the spinor is the conjugate of the other. The requirement is equivalent to the condition $U^\dagger = \tau_x U^\intercal \tau_x$. In the operator language, this is the condition to maintain the fermionic commutations relations under the Bogoliubov transformation. One can readily show that any such U is composed of a finite product of the following matrices: τ_x, τ_y, $e^{i\mu\tau_z}$, and $e^{i\pi m}\mathbb{1}$, where $\mu \in \mathbb{R}$ and $m \in \mathbb{Z}$. The actual number of distinct sequences can be reduced through use of the commutations relations between the generators and is ultimately finite.

To proceed further, it is convenient to gauge away the phase of the superconducting order parameter. This will add spacetime gradients of $\theta(\mathbf{r},t)$ to the electromagnetic potential in the kinetic term. A naive transformation, $e^{i\tau_z\theta/2}$, which involves only the phase of the order parameter, leads to multivaluedness in the presence of vortices. To avoid this problem, Anderson [30] suggested using the transformations $e^{-i(\tau_z\mp\mathbb{1})\theta/2}$, resulting in the superfluid velocity appearing as an effective gauge field in either the electron or the hole component of the Hamiltonian. This gauge choice becomes possible when opposite spins are associated with the two components of the Nambu spinor. Franz and Tešanović [31,32] developed the transformation $e^{i(\tau_z+\mathbb{1})\theta_A}e^{i(\tau_z-\mathbb{1})\theta_B}$ for a periodic bipartite vortex lattice, where A and B are the two sublattices. The vortices should be assigned to the subsets in such a way that the effective magnetic field vanishes on average. Physically, a vortex assigned to subset A will be seen by electrons and be invisible to holes, while vortex assigned to subset B will be seen by holes and be invisible to electrons. Inevitably, in both transformations, particle-hole symmetric structure of the spinors cannot be maintained without additional constraints on the ensemble of allowed partitions of θ.

Instead, we suggest the following transformation:

$$U = e^{i\tau_z\theta(\mathbf{r},t)/2}e^{i\gamma(\mathbf{r},t)}, \tag{2}$$

where θ is the phase function and $\gamma = \pi\sum_j \ell_j$ keeps the transformation properly single-valued by supplying the required sign each time the winding number in θ changes as it evolves in space and time. Our transformation is similar in spirit to the Franz-Tešanović transformation, especially as formulated in Ref. [33], but it manifestly preserves the particle-hole symmetry of the action. Upon applying this gauge transformation, two gauge fields appear in the action: the $a_\nu = A_\nu - \partial_\nu\theta/2$ couples only to the kinetic energy terms, with opposite signs for particles and holes, and the $b_\nu = \partial_\nu\gamma$ couples minimally to momentum, both in the kinetic energy and in the pairing term. We note that the b gauge field is associated with the vortex branch cuts and its corresponding current is proportional to the vortex current. After this transformation, we find

$$\mathcal{G}^{-1} = i\partial_t - b_t + \tau_z a_t - \tau_\mu h_\mu(\mathbf{p}-\mathbf{b},\mathbf{a}), \tag{3}$$

where the 3-vector $h(\mathbf{p},\mathbf{a}) = (\Re\Delta(\mathbf{p}), \Im\Delta(\mathbf{p}), \xi_{\mathbf{p}-\tau_z\mathbf{a}})$.

III. EFFECTIVE ACTION

We can now integrate out the fermion fields to find the effective action, $S_{\rm eff} = \frac{i}{2}\mathrm{Tr}\ln\mathcal{G}$, where $\mathrm{Tr}(\cdot)$ stands for $\int d\mathbf{r}dt\langle\mathbf{r},t|\mathrm{tr}(\cdot)|\mathbf{r},t\rangle$ and "tr" is the trace over the Nambu space. A tedious but straightforward calculation yields (see Appendix), to second order in the gauge fields,

$$S_{\rm eff} = \int d\mathbf{r}dt\Big(na_t + \rho_t a_t^2 - \rho_{ij}a_i a_j - \frac{\kappa_a}{8\pi}\varepsilon_{tij}a_t\partial_i a_j + \frac{\kappa_b}{8\pi}\varepsilon_{\lambda\mu\nu}b_\lambda\partial_\mu b_\nu\Big), \tag{4}$$

where $\varepsilon_{\lambda\mu\nu}$ is the antisymmetric tensor and latin indices i,j run over the spatial components. The coefficients appearing in Eq. (4) are found in terms of $g(\mathbf{k}) \equiv h(\mathbf{k},0)$ as follows:

$$n = \frac{1}{8\pi^2}\int d\mathbf{k}\Big(1-\frac{g_z}{|g|}\Big), \tag{5}$$

$$\rho_t = \frac{1}{16\pi^2}\int d\mathbf{k}\frac{g_x^2+g_y^2}{|g|^3}, \tag{6}$$

$$\rho_{ij} = \frac{1}{16\pi^2}\int d\mathbf{k}\Big(1-\frac{g_z}{|g|}\Big)\partial_{k_i}\partial_{k_j}g_z. \tag{7}$$

Note that n is just the superfluid density. The coefficient of the partial CS term for a,

$$\kappa_a = \frac{1}{4\pi}\int\frac{\varepsilon_{i\nu\lambda}g_i\partial_{k_x}g_\nu\partial_{k_y}g_\lambda}{|g|^3}d\mathbf{k}, \tag{8}$$

is nonuniversal and depends on the details of the system. The coefficient of the full CS term for b, on the other hand,

$$\kappa_b = \frac{1}{4\pi}\int\frac{\varepsilon_{\mu\nu\lambda}g_\mu\partial_{k_x}g_\nu\partial_{k_y}g_\lambda}{|g|^3}d\mathbf{k}, \tag{9}$$

is the Pontryagin charge of the field $g_\mu(\mathbf{k})$ and is therefore always an integer. The action in Eq. (4) is our central result.

In the continuum limit, we have $\xi_{\mathbf{k}} = \mathbf{k}^2/2m - \epsilon_F$ and $\Delta(\mathbf{k}) = v(k_x+ik_y)$. Calculating the coefficients in this limit, we find the following values: $n \sim (mv)^2\ln(\frac{\Lambda}{mv^2})$ with Λ an energy cut-off; $\rho_t = m\kappa_a^\infty/4\pi$; and $\rho_{ij} = (n/2m)\delta_{ij}$, which reflects the Galilean invariance in the continuum [34]. The CS coefficients in the continuum limit are

$$\kappa_a^\infty = \Big[1-2\frac{\epsilon_F}{mv^2}\Theta(-\epsilon_F)\Big]^{-1}, \tag{10}$$

$$\kappa_b^\infty = \Theta(\epsilon_F), \tag{11}$$

where Θ is the step function. Note that this extends the results obtained in Refs. [20,21] to the strong pairing regime, $\epsilon_F < 0$.

For comparison, we have also calculated these coefficients for a system on the square lattice. In this case, $\xi_{\mathbf{k}} = \frac{1}{md^2}(2-\cos k_x d - \cos k_y d) - \epsilon_F$ and $\Delta(\mathbf{k}) = \frac{v}{d}(\sin k_x d + i\sin k_y d)$, where d is the lattice spacing. The coefficients $\kappa_{a,b}^{\infty,\rm sq}$ are

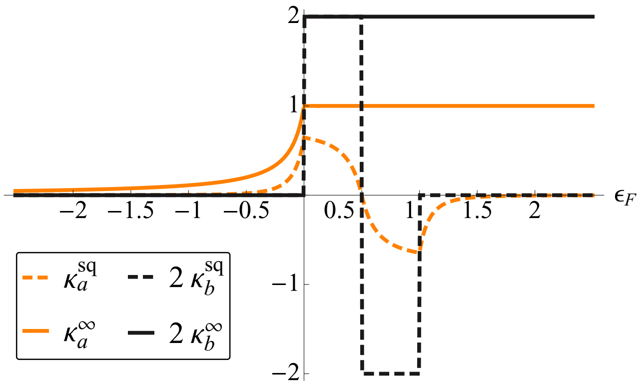

FIG. 1. (Color online) Coefficients of the Chern-Simons terms. The coefficients κ_a and κ_b of the partial (orange) and full (black) Chern-Simons terms are shown for a system in the continuum (solid) and on the square lattice (dashed) for $v = m$. For clarity, we show $2\kappa_b$. The Fermi energy is in units of $4/(md^2)$, the bandwidth of the square lattice, with lattice spacing d. The exchange angles due to the partial and full Chern-Simons terms can be obtained by multiplying the plotted values with $\mp\pi/16$ respectively.

plotted in Fig. 1 as a function of $md^2\epsilon_F/4$. We observe that κ_b^{sq} acquires the values ± 1 in the topological regime $0 < \epsilon_F < 4/(md^2)$ [35] and zero otherwise. In contrast, κ_a^{∞} and κ_a^{sq} are clearly nonuniversal and vary with ϵ_F, showing derivative discontinuities when crossing into the topological regime. The sign change of κ_a^{sq} on the lattice signals a sign reversal in the Hall response of the superconductor [21,36].

IV. VORTEX DYNAMICS AND EXCHANGE

The effective action, Eq. (4), now captures correctly the physics of vortices. This is exemplified by the physical significance of each term appearing in the action. The first term gives rise to the Magnus force on a moving vortex. To see this, note that for a moving vortex $\partial_t\theta = -\dot{\mathbf{x}}\cdot\nabla\theta$, where $\mathbf{x}(t)$ is the position of the vortex. So, the first term yields $-\int dt\, \dot{\mathbf{x}}\cdot\boldsymbol{\mathcal{A}}_M$ with $\boldsymbol{\mathcal{A}}_M = -\int d\mathbf{r}\, n\nabla\theta/2$. Therefore the vortex is subject to a Lorentz-like force $\dot{\mathbf{x}}\times\boldsymbol{\mathcal{B}}_M$ where the Magnus flux $\boldsymbol{\mathcal{B}}_M = \nabla_{\mathbf{x}}\times\boldsymbol{\mathcal{A}}_M = \pi n\hat{\mathbf{z}}$ is proportional to the superfluid density. The contribution from the electromagnetic gauge field A_t in a superconductor vanishes due to the overall charge neutrality of the system [37]. The second and third terms, in conjunction with the Maxwell Lagrangian, give rise to the usual screening of vortices through the Meissner effect. The second term also contributes to the mass of the vortex by generating a term $\int dt \frac{1}{2}m_v\dot{\mathbf{x}}^2$ in the action, where $m_v = \int d\mathbf{r}\rho_t(\nabla\theta/2 - \mathbf{A})^2$.

The fourth and fifth terms, as we now show, carry significant information about the dynamics of vortices. Previous work on the effective low-energy theory of the p-wave superconductor, using only the $\mathbb{U}(1)$ part of our transformation [18,20], yielded an action similar to that of an s-wave superconductor but with an additional partial CS term. Stone and Roy [21] attributed this partial CS term to the existence of a Hall-like response to external fields. They recognized that the Hall current depends on the external field primarily through its effect in modifying the density. Note that the partial CS term we derive here is different from the one appearing in the literature, since in our case $\nabla\times\nabla\theta$ is explicitly nonzero due to the presence of vorticity in θ. Moreover, the full CS term derived here is entirely absent in previous work. As we show now, both of these terms have significant contributions to the exchange statistics of vortices.

Vortices in chiral p-wave superconductors are known to obey non-Abelian statistics, the mechanism behind which relates to the Majorana zero modes localized at their cores. In the presence of $2n$ vortices, the ground state of the system is 2^n-fold degenerate. This degenerate ground state is further divided into two sectors of definite parity $(-1)^N = \pm 1$, where N is the total fermion number. The full braid statistics of vortices can be written using three matrices: R, F, and B. Roughly speaking, R specifies the exchange of two vortices when their fusion outcome is known, F specifies the associativity of the exchange among three anyons, and $B = F^{-1}RF$ is the generator of the full braid group of the vortices in the model. The possible choices of R and F are constrained by consistency relationships. These matrices have been computed for a chiral p-wave superconductor by Ivanov [13] and found to be, up to an overall phase, proportional to those in the Ising anyon model. In this model, a vortex, σ, and antivortex, $\overline{\sigma} = \sigma$, fuse according to the fusion rule $\sigma\times\sigma = I + \psi$, where the fusion channels I and ψ are, respectively, the vacuum (boson) and fermion. In this basis, the F-matrix is real and is given by $F = \frac{1}{\sqrt{2}}(\sigma_z + \sigma_x)$, while $R = e^{-i\chi}\,\text{diag}(1,i)$. The phase $\chi = \pi/8$ is fixed in this model by consistency relations between R and F [38]. However, without a full calculation of χ in the chiral p-wave superconductor, one cannot make a meaningful connection to the Ising anyon model.

Our strategy in this work is to calculate χ by performing a monodromy, which describes a full encircling of one vortex around the other. A general argument shows that the monodromy in the vacuum fusion channel is $R^2 = e^{-2i\chi}$ [38]. This calculation may be done in the same ground state without complications due to the ground-state degeneracy. In our field theory, the monodromy is the Berry's phase in the matrix element of the evolution operator for the exchange of two vortices in the even-parity ground state [39]. See Fig. 2 for illustration.

At first sight, the $\mathbb{Z}_2$ nature of the b gauge field in our effective theory seems to make the calculation of the Berry's phase due to the full CS term tricky. However, this situation is similar to the situation encountered in the singular string gauge of the more common $\mathbb{U}(1)$ gauge theory, in which the gauge field is zero everywhere except on a string emanating from the vortex. One may show that the string gauge is continuously connected to a smooth gauge without changing the winding numbers along the process. Therefore we can calculate the Berry's phase contribution of the b gauge field in the usual way by writing $b = b_1 + b_2$, where b_1 and b_2 are associated with the two vortices, and considering the cross terms between them. Both cross terms contribute equally since, by partial integration, $\int \varepsilon_{\lambda\mu\nu}b_{1\lambda}\partial_\mu b_{2\nu} = \int \varepsilon_{\lambda\mu\nu}b_{2\lambda}\partial_\mu b_{1\nu}$. Assuming for simplicity that only vortex 2 is moving, we have $\varepsilon_{\lambda\mu\nu}\partial_\mu b_{1\nu} = \pi\delta(\mathbf{r})\delta_t^\lambda$, and

$$\chi_b = \frac{\kappa_b}{4}\int d\mathbf{r}dt\;\delta(\mathbf{r})b_{2t} = \frac{\pi\kappa_b}{8}, \tag{12}$$

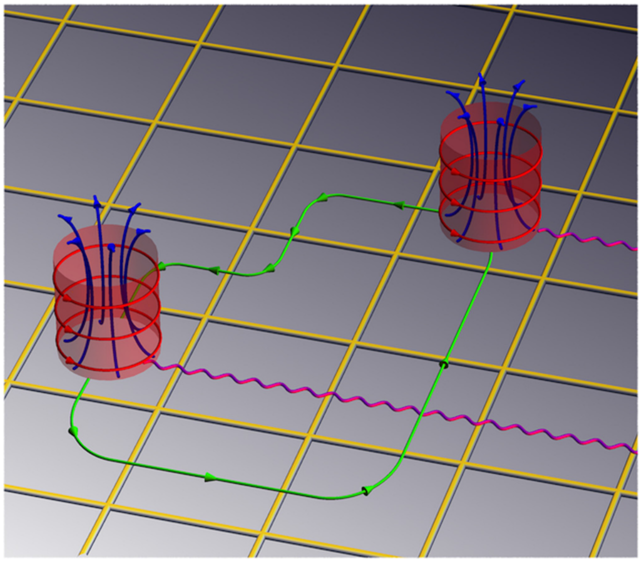

FIG. 2. (Color online) The exchange scheme of two vortices. The exchange path is shown in green and the branch cuts in purple. In a charged superfluid, the magnetic field, blue, is screened by the supercurrent, red. The calculation is simplified when the path is a semicircle of one vortex around the other followed by two radial displacements.

which is, as advertised, quantized in the weak pairing (topological) regime to the value $\pi/8$.

The partial CS term in (4) also contributes to the Berry's phase, albeit not in a quantized fashion due to the nonuniversal behavior of κ_a. We write again $a = a_1 + a_2$ for two vortices and consider the cross terms in the CS term between a_1 and a_2. In a superfluid, the external electromagnetic gauge field is absent and we have $a_{1\mu} = -\frac{1}{2}\partial_\mu \arg(\mathbf{r} - \mathbf{x}_1(t))$, where $\mathbf{x}_1(t)$ is the position of vortex 1, and similarly for a_2. The calculation is simplified by assuming that only vortex 2 moves, so that $a_{1t} = 0$. Then, only one of the cross terms contributes and

$$\chi_a^{\rm sf} = -\frac{\kappa_a}{8\pi}\int d\mathbf{r}dt\; a_{2t}(\nabla \times \mathbf{a}_1)_t = -\frac{\pi\kappa_a}{16}. \qquad (13)$$

In a neutral superfluid, this leads to a nonuniversal long-range contribution to the exchange phase of vortices.

By contrast, for a charged superfluid, the screened magnetic field is screened as $(\nabla \times \mathbf{A})_t = \hat{\mathbf{z}}\, K_0(r/\lambda)/(2\lambda^2)$, associated with a vortex at the origin, where K is the modified Bessel function of the second kind and λ is the (effective) penetration depth. This modifies the result by a geometric phase,

$$\frac{\kappa_a}{8\pi}\int d\mathbf{r}dt\; a_{2t}(\nabla \times \mathbf{A}_1)_t = -\frac{\pi\kappa_a}{16}\left[1 - \frac{R_0}{\lambda}K_1\left(\frac{R_0}{\lambda}\right)\right], \qquad (14)$$

for a circular exchange at distance R_0. So, in a superconductor the total exchange angle due the partial CS term is

$$\chi_a^{\rm sc} = -\frac{\pi\kappa_a R_0}{16\lambda}K_1\left(\frac{R_0}{\lambda}\right). \qquad (15)$$

When the distance between the vortices is much longer than λ this exchange angle vanishes exactly. However, at distances smaller or comparable to λ, nonuniversal contributions to the exchange phase will occur.

Therefore the total exchange angle $\chi = \chi_a + \chi_b$ depends on the details of the dispersion and, in particular, is different in a chiral p-wave superfluid from that in a superconductor due to screening effects.

V. DISCUSSION

In the derivation above, we concentrated on the case where the size of the core of the vortex is vanishingly smaller compared to other length scales. We note that even in this limit, the 2π winding of the phase entails the presence of the protected zero mode in the topological phase. In addition, higher-energy subgap states may occur, localized at the vortex core [40]. The field theory presented above would include the effects of both the zero mode and the subgap states in the two gauge fields a and b if all orders of the loop expansion are retained. To the second order, we find only the Chern-Simons term, which fully encodes the topological exchange phase. This phase is quantized and cannot be modified without closure of the gap. Nonuniversal effects associated with the subgap states may occur in higher order in perturbation theory, which may include effects such as population transfer between closely separated intravortex states [41].

Taking a finite core size may allow additional localized subgap states to get trapped within the vortex. One can model this case by varying the chemical potential ϵ_F around the vortex through the topological phase transition between the topological weak-pairing phase ($\epsilon_F > 0$) outside of the vortex core and the nontopological strong-pairing phase ($\epsilon_F < 0$) within the vortex core. As far as topological properties are concerned, this is equivalent to taking the order parameter to zero at the vortex core but lends itself better to field theoretical analysis [12,42]. In this description, the loci of $\epsilon_F = 0$ that encircle the vortices' cores are internal edges of the system and accommodate gapless excitations. Although their proximity to the Majorana zero modes may affect the coherence of the vortices [41,43,44], as long as they do not mix with the continuum states the Majorana zero mode remains intact [45]. To incorporate this into our field theory, we take the chemical potential to be $\mu(r) = \epsilon_F + \delta\mu(r)$, where $\delta\mu(r)$ denotes the deviation from ϵ_F and has support mostly within the vortex core, i.e., within the coherence length. The new term can be absorbed into $a_t \to a_t - \delta\mu$. One new term that appears in the field theory, $\delta\mu(r)n$, pushes vortices to diffuse along the chemical potential gradient occurring due to other vortices when their cores overlap. A second term $\propto \delta\mu(r)\epsilon_{ij}\partial_i a_j$ generates energetic contributions which go to zero at distances that are larger than the coherence or penetration lengths (whichever is larger). Since there is no coupling between a and b in the effective theory, this modification does not change the topological CS term. At distances larger than the coherence length and to second order in perturbation theory, we find no contribution to the topological Abelian phase associated with the exchange of vortices.

It is also illuminating to compare our results to the non-Abelian Moore-Read state, which is one of the prominent candidate wave functions describing the quantum Hall plateau at filling factor 5/2 [10,12]. While lying in the same universality class of Ising anyons as chiral p-wave superfluids, the Moore-Read state is realized at large magnetic fields, leading to the appearance of an additional Chern-Simons term in the mean-field action. This extra term endows the quasiparticles with an $e/4$ charge and half a flux quantum of fictitious magnetic field. Consequently, there is an additional $\pi/8$ exchange phase for quasiparticles, which should be added to the pure Ising $\pi/8$ contribution, for a total exchange phase of $\pi/4$. Nonuniversal deviations from this value must be governed by the magnetic length, so any mapping of our results to the Moore-Read state, if it at all exists, remains to be worked out. In contrast to the Moore-Read state, our main result here demonstrates that vortices in a screened chiral p-wave superconductor could realize a pristine Ising anyon model.

VI. CONCLUSION

We have derived an effective action of vortices in a spinless chiral p-wave superfluid by properly treating the vortex branch cuts and revealed an Abelian $\mathbb{Z}_2$ gauge structure for the chiral p-wave superfluid. In principle, our transformation is applicable to any pairing symmetry and arbitrary distribution of vortices. In the s-wave case, we have checked that this does not produce additional terms in the action. In the d-wave case, a similar approach has been used to formulate an effective theory of cuprate superconductors [46,47], but no CS term was found.

The topological quantum computation scheme relies on adiabatic braiding of non-Abelian anyons to generate the quantum computation. Among non-Abelian anyon models, Majorana fermions are arguably the closest to experimental work. However, braiding of vortices carrying Majorana fermions is nonuniversal unless supplemented by a missing $\pi/8$ gate. While this gate can be generated by sacrificing topological protection it remains of fundamental importance to provide a proof-of-principle topological scheme to supply the missing $\pi/8$ gate, thus avoiding costly error protection protocols. The results presented here allow the realization of the missing $\pi/8$ gate through multiple braiding of the anyons [17]. As argued above, such braidings should be performed at distances larger than both the coherence length and the screening length.

In this work, we restricted our attention to the Abelian gauge transformations (2). This is enough to infer the Abelian exchange phase of vortices. It can also be used to deduce the existence of zero energy Majorana modes. Using the particle-hole symmetry of the Hamiltonian density, we can write the number density of zero modes as $\nu_0 = 2\langle\bar{\eta}(r)\eta(r)\rangle$ [48,49]. Now, since

$$\langle\bar{\eta}(r)\eta(r)\rangle = 2\langle\delta S_{\text{eff}}/\delta b_t\rangle = \frac{\kappa_b}{2\pi}(\nabla\times\mathbf{b})_t,$$

and b is defined as a $\mathbb{Z}_2$ gauge field, we find

$$\nu_0 = \kappa_b \sum_j \delta(\mathbf{r}-\mathbf{x}_j(t)) \pmod 2, \tag{16}$$

which is quantized and equal to the single winding vortex density (mod 2) in the weak pairing regime. A natural question for future work is whether the other parts of the full group of gauge transformations harbor additional physics. Indeed, as is well known, the zero-energy Majorana modes endow the vortices with the non-Abelian statistics of Ising anyons [12,13]. It would be interesting to see if such a non-Abelian representation emerges in the gauge structure of the effective vortex action by using the entire group of gauge transformations.

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APPENDIX: DERIVATION OF THE ACTION

In the following, we provide the details of the derivation of the effective action appearing in the Results section, Eq. (4). We first write the "dressed" Green's function as $\mathcal{G}^{-1} = \mathcal{G}_0^{-1} + V$, where the "bare" Green's function is $\mathcal{G}_0(k) = (k_t\mathbb{1} - g(\mathbf{k})\cdot\tau)^{-1}$ and V depends explicitly on a and b, i.e.,

$$V = -b_t + \tau_z a_t - \tau_\mu h_\mu(\mathbf{p}-\mathbf{b},\mathbf{a}) + \tau_\mu h_\mu(\mathbf{p},0). \tag{A1}$$

We now perform a perturbative expansion to second order in the gauge fields, writing

$$\begin{aligned} S_{\text{eff}} &= -i\ln \text{Pf}(\mathcal{G}^{-1}) = -i\frac{1}{2}\text{Tr}\ln(\mathcal{G}^{-1}) \\ &= -\frac{i}{2}\text{Tr}\ln\left(\mathcal{G}_0^{-1}\right) - \frac{i}{2}\text{Tr}\ln(1+\mathcal{G}_0 V) \\ &\simeq -\frac{i}{2}\text{Tr}\ln\left(\mathcal{G}_0^{-1}\right) - \frac{i}{2}\text{Tr}(\mathcal{G}_0 V) + \frac{i}{4}\text{Tr}(\mathcal{G}_0 V\mathcal{G}_0 V). \end{aligned} \tag{A2}$$

The fields a and b couple via their associated currents

$$\begin{aligned} j_a^\mu &= \delta_t^\mu\tau_z + \partial_{k_\mu} g_z\left(1-\delta_t^\mu\right), \\ j_b^\mu &= \partial_{k_\mu}\mathcal{G}_0^{-1}. \end{aligned} \tag{A3}$$

To calculate traces, we use the following formulas:

$$\begin{aligned} \text{tr}\{\tau_\mu\tau_\nu\} &= 2\delta_{\mu\nu}, \\ \text{tr}\{\tau_\lambda\tau_\mu\tau_\nu\} &= 2i\epsilon_{\lambda\mu\nu}, \\ \text{tr}\{\tau_\lambda\tau_\mu\tau_\nu\tau_\sigma\} &= 2(\delta_{\lambda\mu}\delta_{\nu\sigma} - \delta_{\lambda\nu}\delta_{\mu\sigma} + \delta_{\lambda\sigma}\delta_{\mu\nu}). \end{aligned} \tag{A4}$$

1. The nonvanishing terms

We proceed to derive the coefficients of the five terms appearing in the action, Eq. (4).

The coefficient $\boldsymbol{n}$. The coefficient multiplying a_t is $n = -\frac{i}{2(2\pi)^3}\int d^3k\,\text{tr}(\mathcal{G}_0\tau_z)$. Since it contains an integration over a single Green's function, care should be taken in its calculation. The correct analytical structure requires that the Green's function is multiplied by an exponent $e^{i\tau_z k_t\eta}$, where $\eta\to 0$, leading to the expression

$$n = -\frac{i}{2(2\pi)^3}\sum_{s=\pm}\int d^3k\frac{sk_t+g_z}{k_t^2-|g|^2+i\eta}e^{is\eta k_t}. \tag{A5}$$

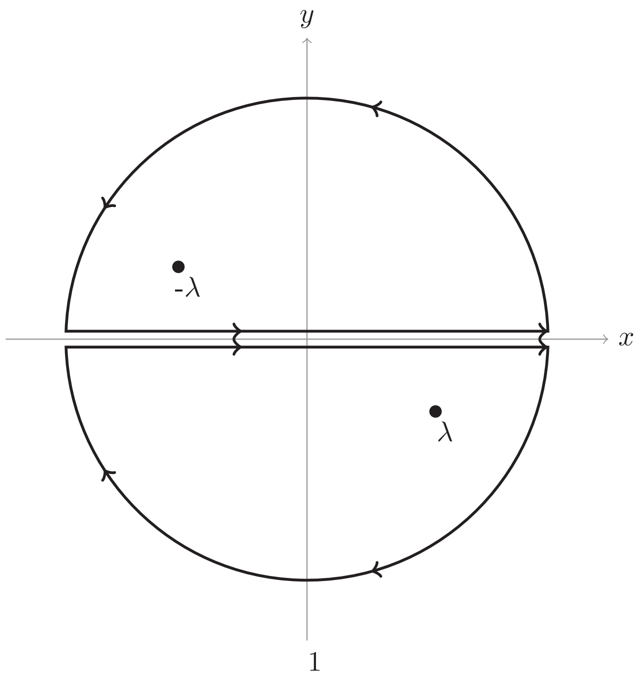

FIG. 3. Contours of integration and poles of the Green's function. The contours of integration that are used in zero temperature calculations for the particle (top) and hole (bottom) part of the Green's function.

Using contour integration over k_t (see Fig. 3 where $\lambda \equiv \sqrt{|g|^2 - i\eta}$), one obtains the expression

$$n = \frac{1}{8\pi^2}\int d\mathbf{k}\left(1 - \frac{g_z}{|g|}\right). \tag{A6}$$

In the p-wave case, the integral is formally divergent and an energy cutoff $\Lambda = \Lambda_k^2/(2m) - \epsilon_F$ needs to be introduced (here Λ_k is a momentum cutoff set only by the inverse lattice spacing)

$$n(\Lambda) = \frac{m}{4\pi}\int_{-\epsilon_F}^{+\Lambda} d\xi\left(1 - \frac{\xi}{\sqrt{\xi^2 + 2mv^2(\xi + \epsilon_F)}}\right). \tag{A7}$$

The coefficient $\boldsymbol{\rho_t}$. Writing the appropriate second order correlator,

$$\begin{aligned}\rho_t &= \frac{i}{32\pi^3}\int d^3k\ \mathrm{tr}\big(\mathcal{G}_0 j_a^t \mathcal{G}_0 j_a^t\big)\\ &= \frac{i}{16\pi^3}\int d^3k \frac{k_t^2 - g_x^2 - g_y^2 + g_z^2}{\big(k_t^2 - |g|^2 + i\eta\big)^2}\\ &= \frac{1}{16\pi^2}\int d\mathbf{k}\frac{g_x^2 + g_y^2}{|g|^3},\end{aligned}$$

where we used the integral

$$\int_{-\infty}^{\infty} dk_t \frac{\alpha k_t^2 + \beta}{\big(k_t^2 - |g|^2 + i\eta\big)^2} = \frac{i\pi(-\alpha|g|^2 + \beta)}{2|g|^3}, \tag{A8}$$

with $\alpha = 1$ and $\beta = g_z^2 - g_x^2 - g_y^2$. For p-wave superfluids in the infinite system limit,

$$\rho_t = \frac{1}{16\pi^2}\int d\mathbf{k}\frac{v^2\mathbf{k}^2}{\big(\xi_{\mathbf{k}}^2 + v^2\mathbf{k}^2\big)^{3/2}} = \frac{m\kappa_a^\infty}{4\pi}, \tag{A9}$$

where $\kappa_a^\infty = (1 - \frac{\epsilon_F - |\epsilon_F|}{mv^2})^{-1}$ coincides with the coefficient of the partial CS term, to be derived below.

The coefficient $\boldsymbol{\rho_{ij}}$. Formally, this coefficient has contributions both from first order and second order in the gradient expansion. The first-order contribution is

$$\frac{-i}{2(2\pi)^3}\int d^3k\ \mathrm{tr}\left[\mathcal{G}_0\left(-\frac{\tau_z}{2m}\delta_{ij}\right)\right] = -\frac{n}{2m}\delta_{ij}. \tag{A10}$$

For $g_z = \xi$, we can write $\delta_{ij}/m = \partial_{k_i}\partial_{k_j} g_z$, to obtain the form in the main text. The second-order contribution exactly vanishes following the integration over k_t,

$$\begin{aligned}&\frac{i}{32\pi^3}\int d^3k\ \mathrm{tr}\big(\mathcal{G}_0 j_a^i \mathcal{G}_0 j_a^j\big)\\ &\quad= \frac{i}{16\pi^3}\int d^3k \frac{\partial g_z}{\partial k_i}\frac{\partial g_z}{\partial k_j}\frac{k_t^2 + |g|^2}{\big(k_t^2 - |g|^2 + i\eta\big)^2} = 0.\end{aligned} \tag{A11}$$

The coefficient $\boldsymbol{\kappa_a}$. To calculate κ_a, we consider the correlator of j_a^t and j_a^j to first order in q_i (no summation convention)

$$\begin{aligned}&\frac{iq_i}{64\pi^3}\int d^3k\ \mathrm{tr}\left(\partial_{k_i}\mathcal{G}_0\tau_3\mathcal{G}_0\frac{\partial g_z}{\partial k_j} - \mathcal{G}_0\tau_3\partial_{k_i}\mathcal{G}_0\frac{\partial g_z}{\partial k_j}\right)\\ &\quad= \frac{iq_i}{64\pi^3}\int d^3k\ \mathrm{tr}\left([\partial_{k_i}\mathcal{G}_0,\tau_3]\mathcal{G}_0\frac{\partial g_z}{\partial k_j}\right)\\ &\quad= \frac{-q_i}{16\pi^3}\sum_{\ell m}\int d^3k\frac{1}{\big(k_t^2 - |g|^2 + i\eta\big)^2}\epsilon_{\ell m}g_\ell\frac{\partial g_m}{\partial k_i}\frac{\partial g_z}{k_j}\\ &\quad= \frac{-iq_i}{32\pi^2}\sum_{\ell m}\int d\mathbf{k}\frac{1}{|g|^3}\epsilon_{\ell m}g_\ell\frac{\partial g_m}{\partial k_i}\frac{\partial g_z}{\partial k_j}.\end{aligned} \tag{A12}$$

For the infinite system p-wave superfluid this results in (no summation convention)

$$\frac{iq_i\epsilon_{ij}}{32m\pi^2}\int d\mathbf{k}\frac{v^2k_j^2}{(\xi_{\mathbf{k}}^2 + v^2\mathbf{k}^2)^{3/2}} = \frac{iq_i\epsilon_{ij}}{16\pi}\kappa_a^\infty. \tag{A13}$$

The coefficient $\boldsymbol{\kappa_b}$. For convenience, we consider one of the correlators giving rise to the CS coefficient

$$\begin{aligned}&\frac{iq_t}{64\pi^3}\int d^3k\ \mathrm{tr}[\partial_{k_t}\mathcal{G}_0(\partial_{k_x}g\ \tau)\mathcal{G}_0(\partial_{k_y}g\ \tau)\\ &\qquad - \mathcal{G}_0(\partial_{k_x}g\cdot\tau)\partial_{k_t}\mathcal{G}_0(\partial_{k_y}g\ \tau)]\\ &\quad= \frac{iq_t}{32\pi^2}\int d\mathbf{k}\frac{\epsilon_{\mu\nu\lambda}g_\mu\partial_{k_x}g_\nu\partial_{k_y}g_\lambda}{|g|^3}.\end{aligned} \tag{A14}$$

For the infinite system p-wave superfluid, we get

$$\kappa_b^\infty = \frac{1}{4\pi}\int d\mathbf{k}\frac{\big(\frac{\mathbf{k}^2}{2m} + \epsilon_F\big)v^2}{\big[v^2\mathbf{k}^2 + \big(\frac{\mathbf{k}^2}{2m} - \epsilon_F\big)^2\big]^{3/2}} = \Theta(\epsilon_F). \tag{A15}$$

2. The vanishing terms

We provide an argument for the decoupling of the a and b fields, as well as for the vanishing of all mass terms for the field b.

The decoupling of the fields $\boldsymbol{a}$ *and* $\boldsymbol{b}$. It can be shown that the integrand of the correlator describing the coupling between a and b,

$$\int d^3k\,\mathrm{tr}\left[\mathcal{G}_0\left(k+\frac{q}{2}\right)j_a^\mu(\mathbf{k})\mathcal{G}_0\left(k-\frac{q}{2}\right)j_b^\nu(\mathbf{k})\right], \qquad \text{(A16)}$$

is always odd under $k \to -k$. Therefore it vanishes to all orders in q following an integration over k.

The absence of mass terms for the field $\boldsymbol{b}$. In first order in the gradient expansion, we find the following contribution to the mass of b:

$$\frac{-i}{2(2\pi)^3}\int d^3k\,\mathrm{tr}\left[\mathcal{G}_0\left(-\frac{\tau_z}{2m}\right)\right] = -\frac{n}{2m}. \qquad \text{(A17)}$$

Another contribution appears in second order (no summation convention),

$$\frac{i}{32\pi^3}\int d^3k\,\mathrm{tr}\left(\mathcal{G}_0 j_b^i \mathcal{G}_0 j_b^i\right) = \frac{1}{16\pi^2}\int d\mathbf{k}\frac{|g|^2(\partial_{k_i}g\,\partial_{k_i}g)-(g\,\partial_{k_i}g)^2}{|g|^3}, \qquad \text{(A18)}$$

where in the infinite system p-wave superfluid we get, after integration over the angle of $\mathbf{k}$,

$$\frac{1}{16\pi}\int_0^{\Lambda_k} d|\mathbf{k}|\frac{v^2|\mathbf{k}|}{|g|^3}\left(\frac{\mathbf{k}^4}{2m^2}+2\epsilon_F^2+v^2\mathbf{k}^2\right). \qquad \text{(A19)}$$

While each is formally divergent, the sum of the two contributions, Eqs. (A17) and (A19), now converges to zero,

$$\lim_{\Lambda_k\to\infty}\left[\frac{n}{2m}-\frac{1}{16\pi}\int d|\mathbf{k}||\mathbf{k}|\frac{v^2}{|g|^3}\left(\frac{\mathbf{k}^4}{2m^2}+2\epsilon_F^2+v^2\mathbf{k}^2\right)\right]$$
$$= \lim_{\Lambda_k\to\infty}\left(\frac{n}{2m}-\frac{\partial_m n}{4}\right)-\frac{|\epsilon_F|+mv^2}{8\pi}\kappa_a^\infty = 0. \qquad \text{(A20)}$$

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2.2 Signatures of the topological spin of Josephson vortices in topological superconductors

Daniel Ariad and Eytan Grosfeld

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Signatures of the topological spin of Josephson vortices in topological superconductors

Daniel Ariad and Eytan Grosfeld
Department of Physics, Ben-Gurion University of the Negev, Be'er-Sheva 84105, Israel

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We consider a modified setup for measuring the Aharonov-Casher phase which consists of a Josephson vortex trapped in an annular topological superconducting junction. The junction encloses both electric charge and magnetic flux. We discover a deviation from the Aharonov-Casher prediction whose origin we identify in an additive universal topological phase that remarkably depends only on the parity of the number of vortices enclosed by the junction. We show that this phase is $\pm 2\pi$ times the topological spin of the Josephson vortex and is proportional to the Chern number. The presence of this phase can be measured through its effect on the junction's voltage characteristics, thus revealing the topological properties of the Josephson vortex and the superconducting state.

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One of the exciting aspects of topological order is the anyonic excitations it supports, which admit fractional charge and exotic quantum statistics. Several fundamental types of anyons can be realized as vortex defects in topological superconductors, generating intensive interest in their properties [1–3]. However, detecting the anyonic properties of these vortices is an ongoing challenge. It has been proposed [4] that Josephson vortices retain the anyonic properties of bulk vortices and thus could be viable candidates for the interference experiments required to unequivocally measure their statistics. However, determining the anyon class of Josephson vortices requires finding the value of their universal exchange phase, which has not yet been reported. This exchange phase is of particular interest as it was argued that it could be used to supplement the set of quantum gates generated by the Josephson vortices to form a universal set [5,6].

In this Rapid Communication, we report a method to calculate the universal exchange phase for Josephson vortices and propose a proof-of-principle experiment by which to measure it. We derive an effective quantum Hamiltonian for a Josephson vortex in a topological Josephson junction [TJJ; see Eq. (12)], unveiling the role of the low-lying Majorana edge states in the soliton dynamics. For the case of a soliton going around an annular Josephson junction [7,8] (see Fig. 1), the soliton accumulates a universal phase related to the exchange phase of Ising anyons. This phase can be exploited to induce a persistent motion of the vortex around the junction, triggered by the nucleation of an additional vortex in the region enclosed by the junction (i.e., by changing the magnetic flux Φ through the central hole). This induced motion drives the Josephson junction into its finite voltage state [9], revealing the presence of the phase.

Our results therefore uncover a significant difference between nontopological Josephson junctions and TJJs. For the former, an externally induced charge Q can drive the Josephson vortex into a persistent motion [7] through the Aharonov-Casher effect [10–12]. This system is analogous to an Aharonov-Bohm ring for electrons. However, the Josephson vortex remains unaffected by other vortices in the system. In contrast, for TJJs, the persistent motion of the Josephson vortex can be controlled with, instead of one knob, two: (i) continuously using the induced charge Q in the region enclosed by the junction and (ii) using the enclosed flux which nucleates vortices inside the path of the vortex, hence changing their parity. In units of electron charge, the nucleation of an extra vortex within the central region is equivalent to an $e/4$ (where e is the electronic charge) shift in the enclosed charge Q.

The dynamics of a TJJ is governed by a modified sine-Gordon Hamiltonian, where the regular bosonic degrees of freedom couple with the low-lying Majorana fermions. In particular, properties of phase solitons (Josephson vortices) through the junction are modified so that each soliton carries a Majorana zero mode [4,13–16]. While experiments to probe the presence of this Majorana mode have been proposed [4,14,17], little attention has been given to the universal properties of the host soliton itself.

We start by discussing the fundamental mechanism behind the topological spin of a Josephson vortex. We then derive explicitly an effective Hamiltonian for the Josephson vortex and demonstrate how the topological spin plays a role in its dynamics. Next, we calculate the Berry connection governing the phase that the Josephson vortex accumulates. Finally, we propose a setup for measuring this phase.

Topological spin of the Josephson vortex. We start by identifying the origin of the topological spin of the Josephson vortex. TJJs [13,18] differ from their nontopological counterparts by the presence of a pair of one-dimensional counterpropagating Majorana states present at the junction, with a Hamiltonian $H_\psi = H + \bar{H}$ (H describes the external edge, and $\bar{H}$ describes the internal one):

$$H = i\frac{v}{2}\int dx\, \psi(x)\partial_x\psi(x),$$
$$\bar{H} = -i\frac{v}{2}\int dx\, \bar{\psi}(x)\partial_x\bar{\psi}(x). \qquad (1)$$

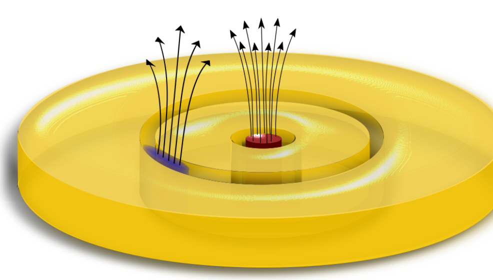

FIG. 1. An annular topological Josephson junction trapping a single soliton. The soliton is depicted in blue. Counterpropagating Majorana edge states are nucleated in the junction. A charge Q and phase Φ are induced externally within the central region (red).

Here $x \in [0,L]$ is the coordinate of the edge, and v is the neutral edge velocity. The fields obey anticommutation relations of the form $\{\psi(x),\psi(x')\} = \{\bar{\psi}(x),\bar{\psi}(x')\} = \delta(x - x')$, and $\{\psi(x),\bar{\psi}(x')\}$=0. We perform the following mode expansion:

$$\psi(x) = \sqrt{\frac{1}{L}} \sum_n e^{-2\pi i n x/L} \psi_n,$$
$$\bar{\psi}(x) = \sqrt{\frac{1}{L}} \sum_n e^{2\pi i n x/L} \bar{\psi}_n. \quad (2)$$

The modes ψ_n satisfy $\{\psi_n,\psi_{n'}\} = \delta_{n+n',0}$ (with similar notation for the opposite chirality). Note that in particular this implies $\psi_0^2 = 1/2$ (for either chirality). Plugging this into the Hamiltonian, we get

$$H = \frac{2\pi v}{L}\left[\frac{1}{2}\sum_n n\psi_{-n}\psi_n\right] \equiv \frac{2\pi v}{L}\mathcal{L},$$
$$\bar{H} = \frac{2\pi v}{L}\left[\frac{1}{2}\sum_n n\bar{\psi}_{-n}\bar{\psi}_n\right] \equiv \frac{2\pi v}{L}\bar{\mathcal{L}}. \quad (3)$$

We now explore the properties of $\mathcal{L}$ and $\bar{\mathcal{L}}$, the dimensionless momentum operators. Using Eq. (2), periodic boundary conditions on the Majorana field imply $n \in \mathbb{Z}$, while antiperiodic boundary conditions imply $n \in \mathbb{Z} + \frac{1}{2}$.

We examine the change in momentum when the boundary conditions are exchanged between periodic and antiperiodic for a closed circular Josephson junction in the absence of tunneling. We write $\mathcal{L}$ and $\bar{\mathcal{L}}$ as

$$\mathcal{L} = \sum_{n>0} n\psi_{-n}\psi_n - \frac{1}{2}\sum_{n>0} n \equiv \sum_{n>0} n\psi_{-n}\psi_n + \mathcal{L}_0(N_v),$$
$$\bar{\mathcal{L}} = \sum_{n>0} n\bar{\psi}_{-n}\bar{\psi}_n - \frac{1}{2}\sum_{n>0} n \equiv \sum_{n>0} n\bar{\psi}_{-n}\bar{\psi}_n + \bar{\mathcal{L}}_0(\bar{N}_v), \quad (4)$$

where $\mathcal{L}_0$ ($\bar{\mathcal{L}}_0$) is the ground-state contribution and N_v ($\bar{N}_v$) denotes the number of vortices enclosed by the external (internal) edge. Specifically, when there is an odd number of vortices enclosed by the edge, $n \in \mathbb{Z}$; otherwise, $n \in \mathbb{Z} + 1/2$. We now calculate the difference in the ground-state contribution in the presence of a Josephson vortex within the junction, i.e., $N_v = 1$ and $\bar{N}_v = 0$. We employ a regularizing function $F(x)$ such that $F'(x) = \partial_x F(x)$ decays to zero faster than $1/x^2$ when $x \to \infty$ and $F'(0) = 1$. We calculate the regularized sum [19]

$$\Delta\mathcal{L}_0 = \mathcal{L}_0(1) - \bar{\mathcal{L}}_0(0)$$
$$= -\frac{1}{2}\sum_{n=1}^{\infty}\left\{nF'(\alpha n) - \left(n-\frac{1}{2}\right)F'\left[\alpha\left(n-\frac{1}{2}\right)\right]\right\}. \quad (5)$$

By taking the limit $\alpha \to 0$ we now get

$$\Delta\mathcal{L}_0 = -\frac{1}{2}\partial_\alpha \sum_{n=1}^{\infty}\left\{F(\alpha n) - F\left[\alpha\left(n-\frac{1}{2}\right)\right]\right\}$$
$$= -\frac{1}{2}\partial_\alpha \sum_{n=1}^{\infty}\left[\frac{\alpha}{2}F'(\alpha n) - \left(\frac{\alpha}{2}\right)^2\frac{1}{2}F''(\alpha n)\right]$$
$$= -\frac{1}{2}\partial_\alpha \int_{\alpha/2}^{\infty} d(\alpha n)\left[\frac{1}{2}F'(\alpha n) - \frac{\alpha}{8}F''(\alpha n)\right]$$
$$= \frac{1}{16}[F'(0) + F'(\infty)] = \frac{1}{16}. \quad (6)$$

This result gives the value of the topological spin of the vortex, which is related to the dimension of the spin operator of the Ising conformal field theory (see, e.g., [20]). In the following we explore how this quantized momentum shift can affect the dynamics of the soliton in the presence of tunneling between the two Majorana edge states.

Effective Hamiltonian for the Josephson vortex. We now proceed to show that the effective description of a Josephson vortex contains explicitly the topological spin discussed above. We turn on the electron tunneling across the junction, leading to a Josephson term and a Majorana tunneling term.

The Josephson term is encapsulated in H_φ, which governs the dynamics of the relative phase degree of freedom φ across the junction [21]:

$$H_\varphi = \frac{\hbar\bar{c}}{g^2}\int dx\left\{\frac{1}{2\bar{c}^2}\dot{\varphi}^2 + \frac{1}{2}\varphi'^2 + \frac{1}{\lambda^2}[1-\cos\varphi]\right\}, \quad (7)$$

where $\dot{\varphi} \equiv (g^2\bar{c}/\hbar)\Pi$, with Π being the canonical momentum, λ is the Josephson penetration length, $\bar{c}$ is the renormalized velocity of light, and g is a dimensionless constant which depends on the parameters of the junction [7].

The Majorana tunneling term is first order in the electron tunneling and takes the form

$$H_{\text{tun}} = i\int dx\, W(x)\psi(x)\bar{\psi}(x), \quad (8)$$

where $W(x) = m\cos[\varphi(x)/2]$ is the Majorana mass term [4,13].

The full Hamiltonian for the TJJ, $H_{\text{TJJ}} = H_\varphi + H_\psi + H_{\text{tun}}$ [4], is an extension of the supersymmetric sine-Gordon model for general values of m [15]. The bosonic degrees of freedom couple with the low-lying Majorana fermions, which we now turn to solve in the presence of a single soliton.

We consider the solution for a classical soliton in the nonrelativistic limit which for short and long Josephson junctions takes the approximate forms [22]

$$\varphi_s(x,q(t)) \simeq 2\pi\left(\frac{x-q(t)}{L}\right), \quad \lambda \gg L,$$
$$\varphi_s(x,q(t)) \simeq 4\arctan\exp\left(\frac{x-q(t)}{\lambda}\right), \quad \lambda \ll L, \quad (9)$$

respectively, with a center-of-mass coordinate at $q(t)$. We plug the solution into the Euclidean action derived from the Hamiltonian H_φ to get the energy associated with the soliton center-of-mass coordinate [23], $\frac{1}{2}m_s\dot{q}^2 + E_0$, where we defined the soliton mass m_s [$m_s = (2\pi)^2\hbar/g^2\bar{c}L$ for $\lambda \gg L$ and $m_s = 8\hbar/g^2\bar{c}\lambda$ for $\lambda \ll L$] and the soliton rest energy [7]. We now proceed to the Majorana sector, $H_\psi = \int dx\, \Psi^T H_0 \Psi$, with $\Psi = (\psi\ \bar{\psi})^T$ and

$$H_0 = \frac{1}{2}\begin{bmatrix} iv\partial_x & iW(x,q(t)) \\ -iW(x,q(t)) & -iv\partial_x \end{bmatrix}, \quad (10)$$

where $W(x,q(t)) = m\cos[\varphi_s(x,q(t))/2]$. The equations simplify considerably by taking a Galilean boost to the moving frame,

$$x' = x - q(t), \quad t' = t,$$
$$\partial_x = \partial_{x'}, \quad \partial_t = -\dot{q}\partial_{x'} + \partial_{t'}.$$

We see that the Majorana fields couple to the center-of-mass velocity of the soliton via a vector-potential-like term that measures the total momentum carried by the two counterpropagating edge states, taking the form

$$\frac{i}{2}\dot{q}\int dx(\psi\partial_x\psi + \bar{\psi}\partial_x\bar{\psi}) = \frac{2\pi}{L}\dot{q}(\mathcal{L} - \bar{\mathcal{L}}). \quad (11)$$

The junction Hamiltonian H_{TJJ}, written in the background of a single soliton, is given in terms of the soliton's center-of-mass momentum $\hat{p}$ (which we now reinstate as a quantum operator) as

$$H_s = E_0 + \frac{1}{2m_s}\left[\hat{p} - \frac{2\pi}{L}(\mathcal{L} - \bar{\mathcal{L}})\right]^2 + \frac{2\pi v}{L}(\mathcal{L} + \bar{\mathcal{L}}) + i\int dx\, W(x)\psi(x)\bar{\psi}(x). \quad (12)$$

This Hamiltonian describes the dynamics of the Josephson vortex within the junction and is our first main result. The ground-state contribution to the vector potential is given by

$$\frac{2\pi}{L}(\mathcal{L}_0 - \bar{\mathcal{L}}_0) = (-1)^{N_v}\frac{2\pi}{L}\frac{1}{16}, \quad (13)$$

coinciding with the one calculated previously in Eq. (6). This suggests that the topological spin of the soliton affects its dynamics and may be measurable. We next turn to show that the low-lying fermion states do not affect the universality of this phase in the adiabatic limit by providing numerical evidence.

The Berry connection. Due to the interactions of the Josephson vortex with the subgap states of energies Δ_n ($n = 0,1,\ldots$), the phase of the soliton is universal only when its traverse time around the junction is large compared to $\hbar/\Delta_1$. We establish this by introducing a numerical procedure for finding the Berry phase that the ground state $|\Omega_q\rangle$ accumulates as function of the position of the soliton, q.

We take a short Josephson junction. When the soliton goes adiabatically around the junction, the Majorana edge states depend parametrically on its position. In addition, there is a $\mathbb{Z}_2$ phase associated with the motion of the soliton: when the soliton completes a cycle, each fermionic mode enclosed by its motion acquires a minus sign. We work in momentum states and truncate the Hilbert space to retain $4N + 2$ modes: $2N$ modes in the antiperiodic edge and $2N + 1$ modes in the periodic edge, the latter including a Majorana zero mode ψ_0. The final mode we retain is the extra Majorana zero-energy state ψ_v, which is localized far from the Josephson junction, either at the center of the annulus or at its outer edge, depending on the parity of the number of vortices in the central hole. In addition, we perform a gauge transformation in which the Majorana fields are single valued under $q \to q + L$ by absorbing the $\mathbb{Z}_2$ phase into the Majorana tunneling term.

Next, we transform the Hamiltonian into a Bogoliubov form for fermions by taking appropriate superpositions of the two zero-energy Majorana fermions, $(\psi_0 \pm i\psi_v)/\sqrt{2}$. The spinor is then rearranged so that particle-hole conjugation is written as $\tau_x K$ (where τ_x is the first Pauli matrix in Bogoliubov space and K is complex conjugation). The Hamiltonian can then be diagonalized via

$$\begin{pmatrix} H_1 & H_2 \\ H_2^\dagger & -H_1^* \end{pmatrix}\begin{pmatrix} U & V^* \\ V & U^* \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}\begin{pmatrix} U & V^* \\ V & U^* \end{pmatrix}. \quad (14)$$

The correct choice of the zero mode that is contained in the positive-energy group of $2N + 1$ eigenvectors leads to a nonvanishing determinant of U. We can then use Eq. (14) to form the BCS ground state $|\Omega_q\rangle$. Explicitly, for $q = 0$, the Hamiltonian blocks are $H_1 = \bigoplus_{k=0}^{2N}\frac{k\pi}{L}$ and

$$H_2 = \frac{m}{2}\left[0 \oplus \left(\bigoplus_{k=1}^{N}\sigma_y\right) - \left(\frac{\sigma_y}{\sqrt{2}}\right) \oplus \left(\bigoplus_{k=1}^{N-1}\sigma_y\right) \oplus 0\right].$$

The Berry connection for $|\Omega_q\rangle$ is given by

$$i\langle\Omega_q|\partial_q\Omega_q\rangle = \frac{i}{4}\text{Tr}\{(1 + gg^\dagger)^{-1}[g'g^\dagger - g(g^\dagger)']\}, \quad (15)$$

where $g = (VU^{-1})^*$ [24]. In addition, we define the translation operator T for the soliton $\chi_q = T_q\chi_0$, with $\chi_q^T = (U_q^T, V_q^T)$. T_q is given explicitly by $T_q = Z_q P_q$, with P_q generating the translation and Z_q generating the $\mathbb{Z}_2$ transformation:

$$P_q = P^{(1)} \oplus P^{(2)},\; P^{(1)} = P^{(2)*} = \bigoplus_{n=1}^{2N+1} e^{(-1)^n(1-n)i\pi q/L},$$

$$Z_q = Z^{(1)} \oplus Z^{(2)},\; Z^{(1)} = Z^{(2)} = \bigoplus_{n=1}^{2N+1}(-1)^{\text{mod}(n,2)\lfloor\frac{q}{L}+\frac{1}{4}\rfloor}.$$

We diagonalize Eq. (14) numerically for $q = 0$, and using T_q we obtain the eigenvectors for any other position of the soliton. We substitute into Eq. (15), performing the derivative symbolically. The result is presented in Fig. 2 with the overlap calculated using the Onishi formula, $|\langle\Omega_{-q}|\Omega_q\rangle| = \sqrt{|\det\chi_{-q}^\dagger\chi_q|}$ [25] for two counterpersisting

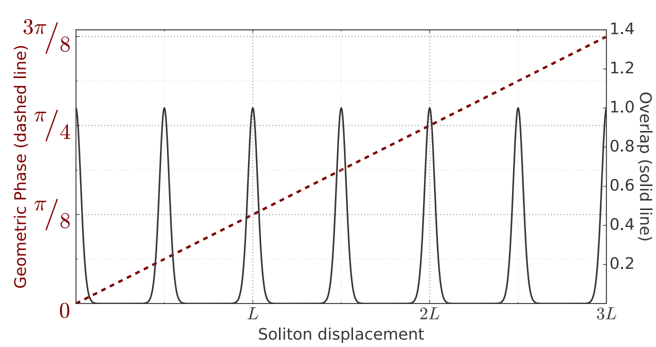

FIG. 2. Numerical results for the geometric phase accumulated by the persisting Josephson vortex. The dashed brown line describes the geometric phase accumulated by each persisting soliton in the presence of a vortex within the central region. In addition, the solid black line describes the overlap norm of two counterpropagating solitons, which becomes nonzero at half cycles. At these points the geometric phase of each soliton acquires its universal values $n\pi/16$, $n \in \mathbb{Z}$.

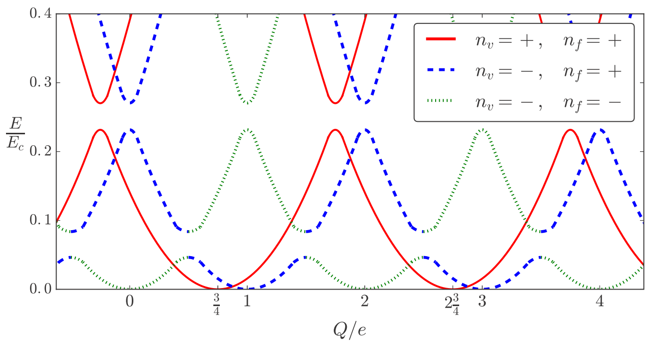

FIG. 3. Energy spectrum for the Josephson vortex. Solid red lines describe the energy of the Josephson vortex in the presence of an even number of vortices enclosed within its path. Dashed blue and dotted green lines describe the case with an odd number of vortices (for even and odd fermion parities, respectively). The velocity of the persisting soliton is proportional to the gradient of the energy, $v_s \propto \partial_Q E$. Fermion parity changing effects open a gap between the green and blue lines, disorder opens a gap between lines of the same color.

solitons, demonstrating that the topological spin is, in principle, an observable. We repeated the procedure taking reversed boundary conditions on the two Majorana edge states, obtaining the same phase but with an additional minus sign, which reproduces Eq. (13) to machine precision.

Proposed setup for detecting the phase shift. We finally consider the setup depicted in Fig. 1 where a single Josephson vortex is trapped within the junction and the voltage between the inner and outer superconducting plates is measured. The energy spectrum of the Josephson vortex can be derived from Eq. (12), and in the presence of an externally induced Aharonov-Casher charge Q within the central region, is given by

$$E_s = E_c\left[\frac{Q}{2e} + \left(\frac{n_f}{4} + \frac{n_v}{16}\right) - N_b\right]^2, \tag{16}$$

where E_c is the charging energy for the junction, $n_f = (-1)^{N_f}$ is the fermion parity within the enclosed path of the Josephson vortex (N_f is the fermion number), $n_v = (-1)^{N_v}$ is the parity of the number of vortices within the same region, and $N_b \in \mathbb{Z}$ is the relative number of Cooper pairs between the two superconducting plates. In the low-energy sector there is an emergent dependence between n_f and n_v: If $n_v = 1$, then $n_f = 1$, but if $n_v = -1$, then n_f is free [4].

Assume we start from the case that there are no vortices within the central hole in the annulus (Fig. 1), i.e., $n_v = 1$ and $n_f = 1$. The junction can be tuned into the zero voltage state by shifting the induced Aharonov-Casher charge Q. The Josephson vortex accordingly acquires a vanishing velocity. Next, we add an extra vortex within the central region of the sample, shifting the value of n_v to -1. The Josephson vortex acquires a phase shift which is equivalent to a $\pm e/4$ shift in the induced Aharonov-Casher charge (see Fig. 3). It then performs a persistent motion, and the junction is driven into its finite voltage state. This dependence of the voltage characteristics of the junction on the number of vortices enclosed within the junction is our second main result.

One possible realization of the system is a topological insulator with an s-wave superconductor deposited on its surface, forming a Josephson junction shaped as in Fig. 1. The dynamics of the soliton will be largely determined by the s-wave superconducting layer, while a Majorana zero mode will be trapped by the soliton on the surface state of the topological insulator. Furthermore, the charge on the central island will be varied by means of a capacitive gate [12].

Discussion. Our central result is the identification of a relative $\pi/4$ phase associated with a Josephson vortex in a topological Josephson junction encircling an odd versus even number of vortices. It is useful to compare this result with the full conformal case which describes the physics with a vanishing Majorana mass, $m = 0$. Then, vortex exchange is captured by a standard fusion rule from conformal field theory (see, e.g., [20]), $\sigma(z)\sigma(0) \sim z^{-1/8}[I + z^{1/2}\psi(z)]$, where I is the identity field and ψ and σ are fields of dimensions $1/2$ and $1/16$, respectively. By identifying the field $\sigma(z)$ as the vortex and $z = x + iy$ as its coordinate, this equation reproduces the presence of a $-\pi/4$ phase shift for a rotation of one vortex around another, $z \to e^{2\pi i}z$. For the case of an odd fermionic number, a $3\pi/4$ phase shift would ensue. Instead, in our case, the nonzero Majorana mass term protects the anyon properties decided by the bulk topological quantum field theory, which is a manifestation of Ocneanu rigidity [26].

Finally, we address the context of this work from experimental and theoretical perspectives. Trapping a single Josephson vortex within an annular Josephson junction has been experimentally achieved [9,27]. It was demonstrated that the Josephson vortex is able to tunnel through a barrier, revealing its quantum nature [9]. Interference experiments of Josephson vortices have been reported [12]. Recently, Josephson vortices were directly observed with scanning tunneling spectroscopy, and their local density of states was deduced [28]. More specifically, in the context of topological superconductors, quasiparticle poisoning may affect observables that are sensitive to fermion parity-changing effects. However, the $e/4$ shift discussed here remains immune to a shift by e, and hence so is the residual motion of the soliton generated by it. Possible realizations of annular topological Josephson junctions were discussed in [4] using semiconductor heterostructures or p-wave superconductors (see, e.g., [29]). Solitons in other scenarios involving p-wave superconductors and two-band superconductors were discussed in [30,31]. Other papers touching on the Aharonov-Casher effect in topological superconductors include [32,33]. The effective action of bulk Abrikosov vortices was considered in [34].

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2.3 How vortex bound states affect the Hall conductivity of a chiral $p \pm ip$ superconductor

Daniel Ariad, Yshai Avishai and Eytan Grosfeld

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How vortex bound states affect the Hall conductivity of a chiral $p \pm ip$ superconductor

Daniel Ariad,[1,*] Yshai Avishai,[1,2,3] and Eytan Grosfeld[1]
[1]*Department of Physics, Ben-Gurion University of the Negev, Beer-Sheva 8410501, Israel*
[2]*NYU-Shanghai University, 1555 Century Avenue, Pudong, Shanghai 200122, China*
[3]*Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*

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The physics of a planar chiral $p \pm ip$ superconductor is studied for various vortex configurations. The occurrence of vortex quasiparticle bound states is exposed together with their ensuing collective properties, such as subgap bands induced by intervortex tunneling. A general method to diagonalize the Hamiltonian of a superconductor in the presence of a vortex lattice is developed that employs only smooth gauge transformations. It renders the Hamiltonian to be periodic (thus allowing the use of the Bloch theorem) and enables the treatment of systems with vortices of finite radii. The pertinent anomalous charge response c_{xy} is calculated (using the Streda formula) and reveals that it contains a quantized contribution. This is attributed to the response to the nucleation of vortices from which we deduce the system's quantum phase.

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I. INTRODUCTION

Measurement of the polar Kerr effect (PKE) in the superconducting state of Sr_2RuO_4 indicates the presence of time-reversal symmetry breaking [1,2]. However, so far no quantitative agreement has been established between theoretical and experimental values of the Kerr angle [3–8]. The latter is proportional to the Hall conductivity, which in turn is proportional to the anomalous charge response c_{xy} [9]. The quantity c_{xy} is finite only in a chiral superconductor [10,11], so the measurement of the PKE provided some of the first evidence for the $p \pm ip$ nature of the order parameter of Sr_2RuO_4.

In this paper, we calculate c_{xy} at zero magnetic field and zero vorticity using a modified Streda formula and show that c_{xy} is a sum of two contributions, one which is nonuniversal, and the other equals $\kappa/8\pi$, where κ is the Chern number of the superconductor, as depicted in Fig. 1. An important insight gained thereby is that an accurate evaluation of c_{xy} requires the knowledge of the charge response to the application of a weak magnetic field and a compensating vortex pair as dictated by imposing periodic boundary conditions (PBCs). This is equivalent to elucidation of the charge response following a chirality flip of the superconductor. Eventually, however, the effect of vortices characteristics (such as their positions as well as their detailed structures) on c_{xy} is minor, and our main results appear to be universal. Once c_{xy} is elucidated, the Hall conductivity at a zero magnetic field and vorticity can be extracted from c_{xy} using a standard procedure [9,11], and that has bearing on the experimentally measured PKE.

In order to substantiate our main result, we need to consider the response of the superconductor to the insertion of a single Dirac flux quanta ($\Phi = h/e$) and compensating pair of vortices. Due to the PBCs imposed on the system when employing the Streda formula, it is natural to solve an equivalent problem for a system composed of many copies of the (originally finite) system, which maps onto an infinite superconductor in the presence of a periodic vortex lattice. The vortices are assumed to have finite radii, thus enabling us to explore the possible dependence of c_{xy} on the presence of vortex bound states.

A natural framework for studying the physics of a periodic vortex lattice is to employ Bloch's theorem. However, this procedure is hindered by the fact that the vector potential and the phase of the order parameter are not independently periodic over the magnetic unit cell (MUC). One may try to apply a gauge transformation to combine the two into a single field which is proportional to the supercurrent. As the latter is periodic in the lattice, Bloch theorem can be employed. However, since the gauge transformation is singular in the presence of vortices, this procedure introduces spurious magnetic fields in the center of the vortices. These spurious

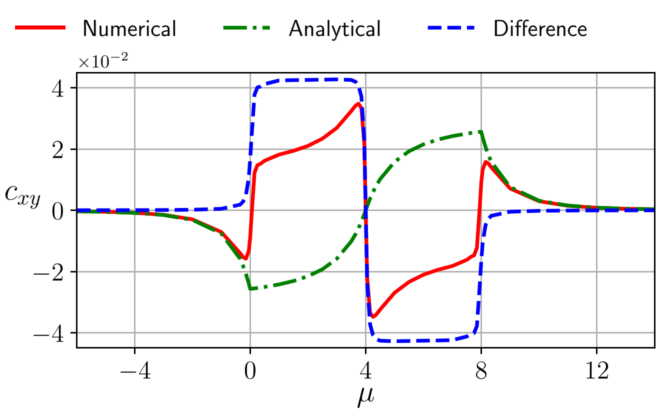

FIG. 1. Average anomalous charge response c_{xy} vs chemical potential μ for a planar p-wave superconductor. The result of a modified Streda formula (Numerical) is compared with the prediction of the effective low-energy theory of a p-wave superconductor (Analytical). Here $t = |\Delta| = 1$ and $\xi = 2.5$. In addition, the magnetic unit cell contains 40×41 sites and two vortices that are pinned on its diagonal, partitioning it in a ratio of 1:2:1.

*daniel@ariad.org

fields either break particle-hole symmetry or introduce branch cuts, originating from the vortex centers, that lead to numerous technical obstacles [12–16].

To circumvent these obstacles, we develop an algorithm to perform an efficient exact diagonalization of the Bogoliubov–de Gennes (BdG) Hamiltonian for an infinite two-dimensional (2D) vortex lattice in a general tight-binding model, that completely avoids the use of singular gauge transformations. Instead, a smooth gauge transformation is employed, that renders both the order parameter and the hopping amplitudes to be independently periodic on the lattice sites.

II. GENERALITIES

It is our perception that the algorithm developed here for the diagonalization of the Hamiltonian is not just a numerical trick, but rather, it meticulously exploits the pertinent physical concepts. Thus, it is worthwhile to illuminate its construction step by step right at the onset. First, we derive an exact expression for the phase of the order parameter by summation over vortices in an ordered array in a superconductor. Second, we transform to another gauge that allows for simultaneously taking the superconducting phase function and the Peierls phases to be periodic functions (mod 2π) without sacrificing any of their properties. Third, we introduce a new gauge for the vector potential, which we dub the "almost antisymmetric gauge (AAG)," which allows accessing, in a system with PBCs, the highest resolution for its magnetic-field dependence. Fourth, we diagonalize the Hamiltonian in a single unit cell under varying boundary conditions per the Bloch theorem, i.e., for different values of the lattice momentum. Thus we extract both the full spectrum of the Hamiltonian and its wave functions.

III. HAMILTONIAN AND ORDER PARAMETER

For spin-1/2 fermions (spin projection $s=\uparrow,\downarrow$), the BdG Hamiltonian in its tight-binding form (taking $\hbar=c=e=1$) consists of three terms $\hat{H}=\hat{T}+\hat{\Delta}-(\mu-4t)\hat{N}$. The hopping term reads

$$\hat{T}=-t\sum_{\boldsymbol{r},s,i}\exp\left(i\int_{\boldsymbol{r}}^{\boldsymbol{r}+\boldsymbol{a}_i}\mathbf{A}\cdot d\boldsymbol{\ell}\right)\psi^{\dagger}_{\boldsymbol{r}+\boldsymbol{a}_i,s}\psi_{\boldsymbol{r},s}+\text{H.c.} \quad (1)$$

The pairing term for an s-wave superconductor is as follows:

$$\hat{\Delta}_{s\text{-wave}}=\sum_{\boldsymbol{r}}\Delta(\mathbf{r})\psi^{\dagger}_{\boldsymbol{r}\uparrow}\psi^{\dagger}_{\boldsymbol{r}\downarrow}+\text{H.c.}, \quad (2)$$

where $\Delta(\boldsymbol{r})=\Delta_0(\boldsymbol{r})\exp(i\Theta(\boldsymbol{r}))$ with $\Delta_0(\boldsymbol{r})$, $\Theta(\boldsymbol{r})$ as real scalar fields and $\boldsymbol{a}_i=a_i\hat{\boldsymbol{\tau}}_i$ (with $i=1,2$) are the lattice vectors. For spinless fermions, we omit one spin component from the hopping term and take the lowest angular momentum p-wave pairing,

$$\hat{\Delta}_{p\text{-wave}}=\sum_{\boldsymbol{r},i}\Delta_{p\pm ip}(\boldsymbol{r},\boldsymbol{a}_i)\psi^{\dagger}_{\boldsymbol{r}\downarrow}\psi^{\dagger}_{\boldsymbol{r}+\boldsymbol{a}_i\downarrow}+\text{H.c.}, \quad (3)$$

where $\Delta_{p\pm ip}(\boldsymbol{r},\boldsymbol{a})=\Delta_0(\boldsymbol{r})\exp(\pm i\,\text{Arg}(\boldsymbol{a}))\exp(i\Theta(\boldsymbol{r}))\exp(\frac{i}{2}\int_{\mathbf{r}}^{\mathbf{r}+\boldsymbol{a}}\nabla\Theta\cdot d\boldsymbol{\ell})$ and $\text{Arg}(\boldsymbol{r})=\text{Arg}(x+iy)$. The superconducting order parameter is defined in such a way that the $\mathbb{U}(1)$ gauge invariance is respected [13].

We recall that vortices are encoded as nodes of the order parameter, characterized by a finite quantized winding number of the phase $\Theta(\boldsymbol{r})$ [17]. In order to form a vortex lattice we tile the plane with a MUC. The MUC is chosen to enclose an even number of vortices. Thus, each vortex within the MUC constitutes a sublattice. The superconducting phase $\Theta(\boldsymbol{r})$ can be written as a sum over contributions of such vortex (or antivortex) sublattices $\Theta(\boldsymbol{r})=\sum_{i=1}^{N_v}s_i\theta(\boldsymbol{r}-\boldsymbol{r}_i)$, where $s_i=+$ ($s_i=-$) for vortices (antivortices) and $\boldsymbol{r}_i$ is the position of the ith sublattice with respect to the origin. Within each sublattice, the phase $\theta(\boldsymbol{r})$ can be expressed by summing the contributions of all vortices in the sublattice,

$$\theta(\boldsymbol{r})=\lim_{M\to\infty}\left[\sum_{m,n=-2M}^{2M}\text{Arg}(\boldsymbol{r}-m\boldsymbol{\tau}_1-n\boldsymbol{\tau}_2)\bmod 2\pi\right], \quad (4)$$

where $\boldsymbol{\tau}_i=q_ia_i\hat{\boldsymbol{\tau}}_i$ are the vectors spanning the MUC, composed of $q_1\times q_2$ atomic sites. Using complex variables $z=x+iy$, we have

$$\theta(z)=\text{Im}\left\{\text{Log}\left[i\vartheta_1\left(\frac{z}{\tau_2},-\frac{\tau_1}{\tau_2}\right)\right]-\frac{2iz^2}{\tau_1\tau_2}\arctan\left(\frac{i\tau_1}{\tau_2}\right)\right\}, \quad (5)$$

where τ_i is the complex representation of the vector $\boldsymbol{\tau}_i$.

It is important to note that, although the resulting function $\theta(\boldsymbol{r})$ admits the correct windings at the positions of the vortices, it is generally nonperiodic on the MUC. Therefore, using this summation for taking PBCs for a single MUC (a torus) is unsafe.

IV. LATTICE PERIODIC GAUGE

We proceed by taking a gauge transformation that renders the order parameter and the hopping amplitudes periodic in the MUC $\mathbf{A}\to\mathbf{A}+\frac{1}{2}\nabla_{\boldsymbol{r}}\chi$, $\Delta\to\Delta\exp(i\chi)$, $\psi_{\boldsymbol{r}s}\to\exp(i\chi/2)\psi_{\boldsymbol{r}s}$. We note that the supercurrent $\mathbf{J}\propto\frac{1}{2}\nabla_{\boldsymbol{r}}\Theta-\mathbf{A}$ is periodic in the two magnetic lattice vectors $\boldsymbol{\tau}_i$ and thus $\int_{\boldsymbol{r}}^{\boldsymbol{r}+\boldsymbol{\tau}_i}\mathbf{J}\cdot\mathbf{d}\boldsymbol{\ell}$ is similarly doubly periodic. Therefore, we can always choose $\chi(\boldsymbol{r})$ so that the fields $\Theta'(\boldsymbol{r})=\Theta(\boldsymbol{r})+\chi(\boldsymbol{r})$ and $\int_{\boldsymbol{r}}^{\boldsymbol{r}+\boldsymbol{\tau}_i}(\mathbf{A}+\frac{1}{2}\nabla_{\boldsymbol{r}}\chi)\cdot\mathbf{d}\boldsymbol{\ell}$ are periodic (mod 2π) on the lattice sites $\boldsymbol{r}_{m,n}=(m/q_1)\boldsymbol{\tau}_1+(n/q_2)\boldsymbol{\tau}_2$. We now show that there exists a gauge that fulfills the conditions above for a MUC composed of $q\times(q+1)$ atomic sites for which $q_2-q_1=1$. For a general vortex lattice, using the same notation as for $\Theta(\boldsymbol{r})$ above, we write $\chi(\boldsymbol{r})=\sum_{i=1}^{N_v}s_i\phi(\boldsymbol{r},\boldsymbol{r}_i)$ where $\phi(\boldsymbol{r},\boldsymbol{r}_i)$ is written in terms of complex variables as

$$\begin{aligned}\phi(z,z_i)=&\,2\,\text{Re}\left[\frac{(z-z_i)^2}{\tau_1\tau_2}\arctan\left(\frac{i\tau_1}{\tau_2}\right)\right]+q\pi\,\text{Re}\left(\frac{z^2}{\tau_1\tau_2}\right)\\&-(q+1)\pi\frac{\text{Im}^2(z/\tau_2)\,\text{Re}(\tau_1/\tau_2)}{\text{Im}^2(\tau_1/\tau_2)}\\&-q\pi\frac{\text{Im}^2(z/\tau_1)\,\text{Re}(\tau_2/\tau_1)}{\text{Im}^2(\tau_2/\tau_1)}+\pi\frac{\text{Im}(z/\tau_1)}{\text{Im}(\tau_2/\tau_1)}\\&+\left[2\pi\,\text{Re}\left(\frac{z_i}{\tau_2}\right)-\pi\right]\frac{\text{Im}(z/\tau_2)}{\text{Im}(\tau_1/\tau_2)}.\end{aligned} \quad (6)$$

The resulting phase function Θ' is now doubly periodic as required. Furthermore, integrating the supercurrent $\mathbf{J}(\boldsymbol{r})$ around the MUC reveals that

$$0 = \oint_{\mathrm{MUC}} \mathbf{J}\cdot \mathbf{d}\boldsymbol{\ell} \propto N_w\Phi_0 - \oint_{\mathrm{MUC}} \mathbf{A}\cdot \mathbf{d}\boldsymbol{\ell}, \tag{7}$$

where $\Phi_0 = h/(2e) = \pi$ is the superconducting magnetic flux quantum and $N_w = \sum_{i=1}^{N_v} s_i$ is the total winding for the vortices in the MUC. Due to the Dirac quantization condition [19], requiring that $\Phi = n(h/e)$ with $n \in \mathbb{Z}$ when taking PBCs on $\boldsymbol{A}$, N_w must be an even number.

V. THE ALMOST ANTISYMMETRIC GAUGE

Our next step is to find a complementary vector field. Due to the periodicity of the supercurrent, the vector potential is required to fulfill the condition,

$$\mathbf{A}(\boldsymbol{r}+\boldsymbol{\tau}_i) = \mathbf{A}(\boldsymbol{r}) + \frac{1}{2}\nabla[\Theta'(\boldsymbol{r}+\boldsymbol{\tau}_i) - \Theta'(\boldsymbol{r})]. \tag{8}$$

We now introduce the AAG that is designed to generate a homogeneous magnetic field and obey Eq. (8) and is given by

$$\mathbf{A} = \frac{2\Phi_0 p}{a_1 a_2 \sin^2(\alpha_1-\alpha_2)}\left[\frac{(\boldsymbol{r}\times\hat{\boldsymbol{\tau}}_1)\times\hat{\boldsymbol{\tau}}_2}{q+1} + \frac{(\boldsymbol{r}\times\hat{\boldsymbol{\tau}}_2)\times\hat{\boldsymbol{\tau}}_1}{q}\right], \tag{9}$$

where $\alpha_i = \mathrm{Arg}\ \tau_i$ and $p \in \mathbb{Z} \bmod q(q+1)$.

The AAG is also useful in other contexts. For example, if one is interested in solving the Hofstadter problem [20] with high-flux resolution, it is obtained by considering a rectangular lattice of size $q \times (q+1)$ and choosing an AAG $\mathbf{A}(\boldsymbol{r}) = 2\Phi_0 p(\frac{y}{q+1}, \frac{x}{q})$ with $p = 1, 2, \ldots, q(q+1)$. The flux per unit cell is then $\frac{2\Phi_0 p}{q(q+1)}$, and thus the flux through the entire 2D system is $2\Phi_0 p$. In the standard procedure using the Landau gauge, the flux through the entire 2D area can only take values from a narrow and sparse range $2\Phi_0 pq$ with $p = 1, 2, \ldots, q+1$.

VI. ELECTRONIC BAND STRUCTURE OF A VORTEX LATTICE

We now elucidate the quasiparticle energy dispersion for the pertinent BdG Hamiltonian, which is depicted in Fig. 2. Consider a vortex lattice made of $N_1 \times N_2$ MUCs with $q_1 \times q_2$ atomic sites in each cell, so in total, the system consists of $L_1 \times L_2$ sites ($L_i = N_i q_i$). The Hamiltonian of the vortex lattice in the BdG representation is written as $\hat{H} = \Psi^\dagger H_{\mathrm{BdG}}\Psi$, where H_{BdG} is the Hamiltonian density. For s-wave superconductors, $\Psi \equiv (\psi_\downarrow, \psi_\uparrow^\dagger)^T$ where ψ_s with $s \in \{\uparrow, \downarrow\}$ is an L_1L_2 component spinor of spin s fermion annihilation operators. For p-wave superconductors, the index s indicates particle and hole subspaces.

Next, we introduce the discrete translation operators along the two lattice directions $i = 1, 2$,

$$T_i \colon \psi_{\boldsymbol{r},s} \longrightarrow \psi_{(\boldsymbol{r}+\boldsymbol{\tau}_i) \bmod N_i\boldsymbol{\tau}_i, s}, \tag{10}$$

which satisfy $[T_1, T_2] = 0$ and $[H_{\mathrm{BdG}}, T_i] = 0$. Clearly, the eigenvalues of T_i are $\exp(i2\pi n_i/N_i)$ with $n_i = 1, 2, \ldots, N_i$.

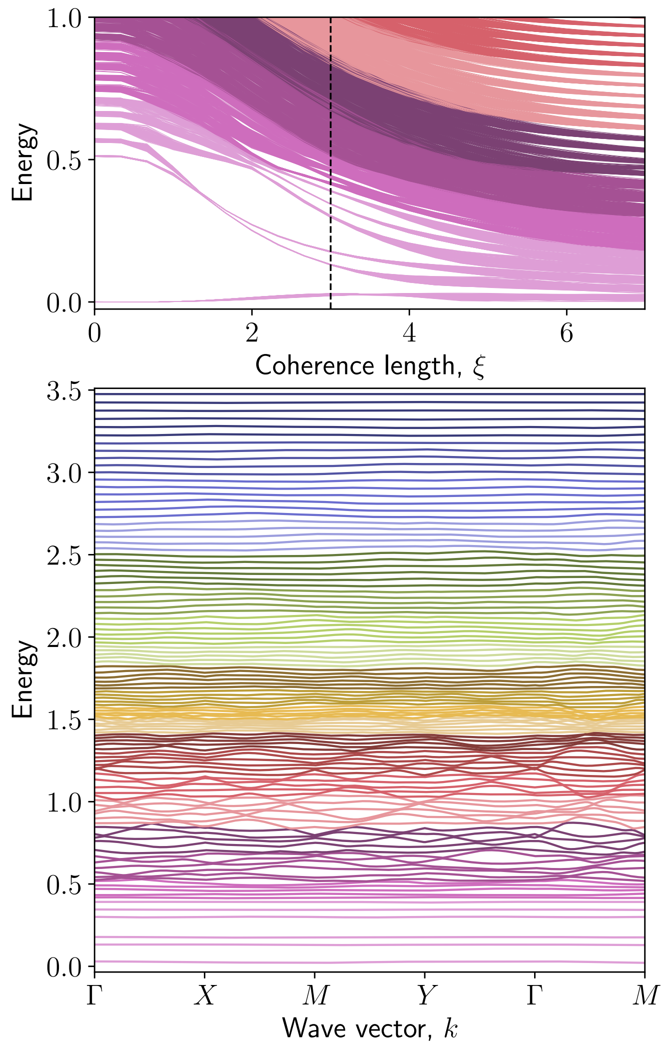

FIG. 2. (Top) Quasiparticle bands as a function of coherence length ξ for a pinned vortex lattice in a p-wave superconductor. The magnetic unit cell contains 10×11 sites and two vortices that are pinned on its diagonal, partitioning it in a ratio of 1:2:1. We take $t = |\Delta| = \mu = 1$. (Bottom) The quasiparticle band structure for $\xi = 2.5$. We observe Landau levels at high energies and Caroli–de Gennes–Matricon states below the gap, including the band generated from zero-mode tunneling [18].

The Bloch theorem is employed by introducing $q_1 \times q_2$ sublattice wave functions,

$$\varphi_{\boldsymbol{k},s}(\boldsymbol{r}) = \frac{1}{\sqrt{N_1N_2}}\sum_{\boldsymbol{R}} \exp(i\boldsymbol{k}\cdot\boldsymbol{R})|\boldsymbol{R}+\boldsymbol{r}, s\rangle, \tag{11}$$

where $\boldsymbol{R} \equiv \boldsymbol{R}_{m_1,m_2} = m_1\boldsymbol{\tau}_1 + m_2\boldsymbol{\tau}_2$ denotes the positions of the MUCs and $\boldsymbol{k} \equiv \boldsymbol{k}_{n_1,n_2} = \frac{2\pi n_1}{N_1|\boldsymbol{\tau}_1|}\hat{\boldsymbol{\tau}}_1 + \frac{2\pi n_2}{N_2|\boldsymbol{\tau}_2|}\hat{\boldsymbol{\tau}}_2$. The Hamiltonian within a given sublattice is defined as

$$H_{\boldsymbol{k}}(\boldsymbol{r}, s; \boldsymbol{r}', s') = \langle\varphi_{\boldsymbol{k},s}(\boldsymbol{r})|H_{\mathrm{BdG}}|\varphi_{\boldsymbol{k},s'}(\boldsymbol{r}')\rangle. \tag{12}$$

In this notation, the particle-hole symmetry of each block takes the form $\Sigma_1 H_{-\boldsymbol{k}}^* \Sigma_1 = -H_{\boldsymbol{k}}$ with $\Sigma_1 = \sigma_1 \otimes I_{q_1q_2}$. The block $H_{\boldsymbol{k}=\boldsymbol{0}}$ corresponds to a single MUC with PBCs. Technically, $H_{\boldsymbol{k}}$ is obtained from $H_{\boldsymbol{0}}$ just by varying the boundary conditions

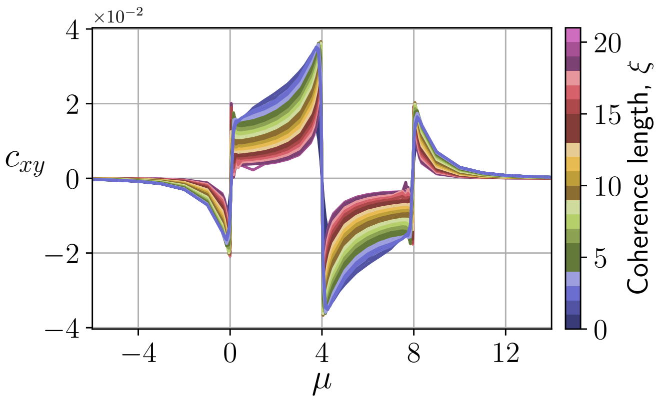

FIG. 3. Average anomalous charge response c_{xy} vs chemical potential μ for different coherence lengths ξ. The p-wave superconductor has a magnetic unit cell of 40×41 sites $t = |\Delta| = 1$. In addition, we pinned two vortices on the magnetic unit cell diagonal, partitioning it in a ratio of 1:2:1.

as follows:

$$H_{\mathbf{0}}(\boldsymbol{r}, s; \boldsymbol{r} + \boldsymbol{\tau}_i, s') \rightarrow H_{\mathbf{0}}(\boldsymbol{r}, s; \boldsymbol{r} + \boldsymbol{\tau}_i, s') \exp(-i\boldsymbol{k} \cdot \boldsymbol{\tau}_i), \tag{13}$$

for any $\boldsymbol{r}$ on the boundary of the MUC.

VII. THE ANOMALOUS CHARGE RESPONSE FUNCTION c_{xy}

In previous studies of bulk p-wave superconductors, it was noted that c_{xy} is not quantized [13,21,22]. We now calculate c_{xy} in the presence of finite-size vortices and discover, remarkably, that c_{xy} contains a universal quantized contribution.

The anomalous charge response is exposed in the effective action of a p-wave superconductor through the appearance of a partial Chern-Simons (pCS) term [4,9],

$$S_{\text{pCS}} = \pm c_{xy} \int d\boldsymbol{r}\, dt\, a_t (\nabla \times \boldsymbol{a})_z, \tag{14}$$

where $a_\mu = A_\mu - \partial_\mu \Theta/2,\ \mu \in \{t, x, y\}$, and the sign corresponds to the superconductor chirality $p_x \pm ip_y$. Thus, in analogy with the Streda formula [23], the following relation holds [13]:

$$c_{xy}(\boldsymbol{r}) = \pm \frac{\partial \rho(\boldsymbol{r})}{\partial B_z}\bigg|_{B_z=0}, \tag{15}$$

where $\rho(\boldsymbol{r}) = \delta S_{\text{eff}}/\delta a_t(\boldsymbol{r}) = \langle \text{gs}| \sum_s \psi^\dagger_{\mathbf{r},s}\psi_{\mathbf{r},s}|\text{gs}\rangle$, $|\text{gs}\rangle$ is the superconducting ground state and $B_z = (\nabla \times \boldsymbol{a})_z$ is homogeneous at the lattice sites. This formula relates the density response to an infinitesimal external magnetic field. However, any variation of the magnetic field imposes a change in the superconducting phase in order to maintain periodicity of the supercurrents. Thus, as we now explain, the physical scenario here requires a modification of the Streda formula. The minimal variation of the magnetic field is a single flux quantum (over the entire system), leading to the nucleation of two vortices. Similarly, when an opposite magnetic field is applied, two antivortices are nucleated. Therefore, the derivative operation in the Streda formula for calculating density response implies a simultaneous flip of magnetic field as well as vortex chiralities. This is equivalent to a chirality flip of the order parameter (from $p_x \pm ip_y$ to $p_x \mp ip_y$). The above procedure is also necessary as two opposite chirality states admit roughly the same spectrum so that the density response can be considered as a small perturbation.

With this insight in mind, it is now possible to use Eq. (15) and numerically calculate the spatial average of $c_{xy}(\boldsymbol{r})$ as a function of μ as shown in Fig. 3. The results are then compared with the analytical expression of c_{xy} from the effective action governing the low-energy dynamics of the p-wave superconductor [13,21,22].

It is found that the two predictions overlap in the trivial phases except that the numerics predict a slight dependence on ξ but not on $|\Delta|$ as shown in Fig. 4. Moreover, in all phases, c_{xy} does not depend on the number of MUCs that form the vortex lattice. Hence, c_{xy} can be calculated from a single MUC corresponding to $\boldsymbol{k} = \mathbf{0}$. Another property of c_{xy} is that its average value within the MUC depends only slightly on its dimensions (as long as the vortices are well separated). Thus, one may expect to obtain c_{xy} for $B_z = 0$ by probing the

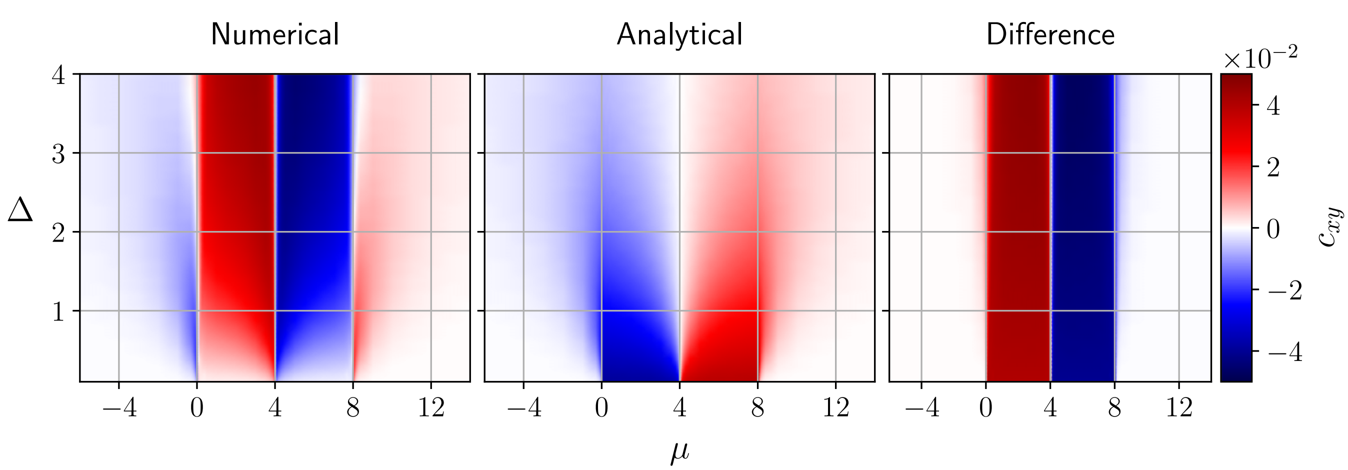

FIG. 4. Average anomalous charge response c_{xy} vs chemical potential μ and order parameter $|\Delta|$ for a p-wave superconductor with a magnetic unit cell of 40×41 sites $t = 1$ and $\xi = 2.5$. The modified Streda formula (Numerical) is compared with the prediction of the effective low-energy theory of the p-wave superconductor (Analytical). In addition, we pinned two vortices on the magnetic unit-cell diagonal, partitioning it in a ratio of 1:2:1.

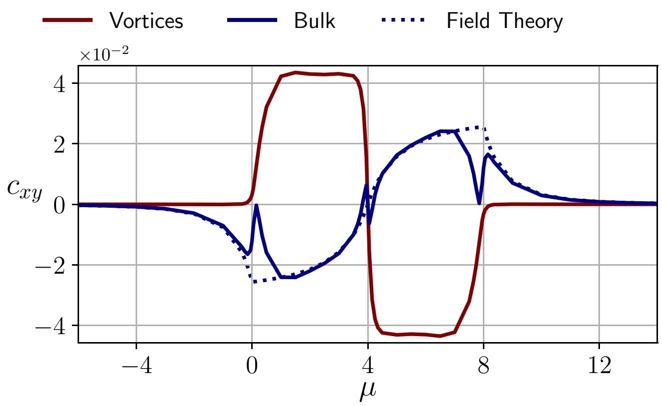

FIG. 5. Average anomalous charge response c_{xy} vs chemical potential μ for a planar p-wave superconductor. The magnetic unit-cell average of c_{xy} is crudely separated into contributions from the *vortices* and contributions from the *bulk*. For comparison, we also present the *field-theory* prediction of c_{xy}. Here $t = |\Delta| = 1$ and $\xi = 2.5$. In addition, the magnetic unit cell contains 40×41 sites and two vortices that are pinned on its diagonal, partitioning it in a ratio of 1:2:1.

density response of a small piece of the superconductor with PBCs for the application of minimal magnetic flux $\Phi = h/e$ and a compensating vortex pair (placed arbitrarily within the superconductor). This is indeed what we observe, and the result matches extremely well with the field-theoretical prediction in the trivial phase. Remarkably, in the topological phases $(0 < \mu < 8)$ there is a sizable discrepancy between our predictions and those based on field theory. Since the charge accumulated at the vortex core (referred to as vortex charging) depends on the angular momentum of the Cooper pairs, it is determined by an interplay among the superconductor chirality, the vorticity, and the quantum phase [24]. We now show that this discrepancy can indeed be traced to a universal vortex charging effect.

To decipher the origin of c_{xy}, we perform two kinds of spatial and spectral cuts. First, we crudely separate the vortex cores at distances $r \leqslant \xi$ from the bulk and average c_{xy} in each region independently to find their respective contributions; in the bulk, both theories yield similar results, whereas at the cores, the numerical results expose steps of $\pm\frac{1}{8\pi}$ as shown in Fig. 5. Second, we separate the charge in the vortices into contributions of each Bogoliubov quasiparticle and take into account those within the energy gap $\Delta Q_{\text{core}} = \iint_{\text{core}} d\boldsymbol{r}\, \Delta\tilde{\rho}_{\boldsymbol{r}}$ with $\tilde{\rho}_{\boldsymbol{r}} = \frac{1}{2}\sum_{0<\epsilon<E_{\text{gap}}}(|v_{\boldsymbol{r},\epsilon}|^2 - |u_{\boldsymbol{r},\epsilon}|^2)$. We then find that the most significant contribution to c_{xy} arises from the Caroli–de Gennes–Matricon states [25]. This demonstrates that the universal contribution to c_{xy} arises from the vortex core and, specifically, from vortex bound states. On the other hand, within the field-theory formalism, the vortices are treated as point singularities, which may explain the discrepancy. Although it was observed in Ref. [24] that vortices with opposite vorticities accumulate different charges, here we show that the relative accumulated charge for opposite vorticities is a universal quantity, which appears to be proportional to the Chern number of the superconductor. For consistency, we checked that s-wave and $d_{x^2-y^2}$-wave superconductors have vanishing anomalous charge responses.

VIII. SUMMARY

In this paper, the nature of the PKE and the order parameter in the $p \pm ip$ superconductor Sr_2RuO_4 is analyzed. A smooth gauge is introduced, that can be used in conjunction with Bloch's theorem to diagonalize BdG Hamiltonians for infinite superconductors in various periodic vortex states. The dispersion of quasiparticle energies for such vortex states with a finite vortex core size is calculated beyond previous numerical studies, and the occurrence of midgap states is demonstrated as the size of the core is increased.

Employing the same diagonalization algorithm, and modifying the Streda formula, the anomalous charge response c_{xy} is calculated in the absence of vortices. The structure of c_{xy} is then used to identify the quantum phases of the pertinent systems. Our results indicate that in p-wave superconductors subjected to PBCs, c_{xy} is calculable by their response to an applied weak magnetic field and the nucleation of a vortex pair. On the other hand, the average value of c_{xy} within the bulk is only weakly affected by the size of the vortices' cores or their positions in the MUC. It is then reasonable to perceive that the discrepancy with results based on the field-theory approach to p-wave superconductors is attributed to vortex charging, which occurs only in vortices with finite core radii.

Finally, it is worth expressing our hope that the AAG introduced here and the ensuing diagonalization algorithm will serve as useful tools in the study of similar systems, such as the Hofstadter butterfly in the presence of disorder [20].

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Chapter 3

Summary and discussion

In this dissertation we derived the effective action of a chiral p-wave superconductor. In our theory the entire vortex contribution is accounted for by gauge fields, in contrast to previous works that used a phenomenological description[51, 50]. In addition, we explicitly evaluated the Abelian part of the exchange matrix of vortices in the 2D chiral spinless p-wave superconductor. We also introduced innovative ways to measure and control solid-state Majorana fermions. Finally, we considered the effect of the vortices on the electromagnetic response of the system, discovering that, when the vortices possess a finite core, the formation of CdGM states and sub-gap bands induced by tunneling between bulk vortices plays an important role. This discovery is highly relevant for modeling and implementing quantum information protocols whose reliability can be examined under various situations, for example, in the presence of disorder and impurities.

*Effective theory of vortices in two-dimensional spinless chiral **p**-wave superfluids*. An effective action depicting the dynamics of vortices in a superfluid was derived from a microscopic theory. More importantly, we demonstrated how to produce the missing CS term, which describes the Abelian part of the statistics of the vortices, in the action. Moreover, this model enables us to predict the conditions under which the Abelian phase will deviate from its universal value. We found that the exchange phase is universal in the chiral p-wave superconductor when screening is present and the distance between vortices is much greater than both the coherence and the penetration lengths, although substantial non-universal deviations occur for a neutral superfluid. *However, the non-Abelian sector, attributed to the zero-modes, is missing from the model and further work is needed to elaborate its origin.*

Signatures of the topological spin of Josephson vortices in topological superconductors. We considered a setup consisting of a Josephson vortex trapped in an annular topological supercondut-ing junction, which encloses both an electrical charge and magnetic flux. The vortex was driven into a persistent motion through an Aharonov-Casher effect. The dynamics of the topological Josephson vortex were governed by a modified Sine-Gordon Hamiltonian, where the regular bosonic degrees of freedom couple with the low lying Majorana fermions. In the limit of vanishing tunneling across the junction, we analyzed the difference in momentum between the inner and outer edges, as the boundary conditions are exchanged between periodic and anti-periodic as depicted in Fig.(3.1). We found that the ground-state contribution to the momentum difference is $\pm\frac{2\pi}{L} \times \frac{1}{16}$, where $\frac{1}{16}$ is the topological spin of the Josephson vortex. Moreover, the universal phase depends solely on the parity of the number of vortices enclosed by the junction. This phase is 2π times the topological spin of the Josephson vortex and is proportional to the Chern number. We proceeded by showing that our results hold true even when tunneling occurs across the junction.

The energy spectrum of the Josephson vortex, in the presence of an externally-induced Aharonov Casher charge Q and N vortices within the central region, was obtained. It revealed that persistent motion of the topological Josephson vortex can be manipulated by two knobs, Q and N mod 2. In contrast, a non-topological Josephson vortex remains unaffected by N. Since both the velocity of the persisting soliton and the voltage across the junction are proportional to the gradient of the energy, $V \propto v_s \propto \partial_Q E$, the topological spin can be measured through its effect on the junction's voltage characteristics. *We note that our platform and the topological spin, in particular, can be exploited to form the sought-after $\frac{\pi}{8}$ magic phase gate, necessary to complete a set of universal quantum gates.*

How vortex bound states affect the Hall conductivity of a chiral $p \pm ip$ superconductor. We presented a systematic way to construct analytically the phase of a complex order parameter for any spatial configuration of vortex defects within a 2D magnetic unit cell with periodic boundary conditions. This order parameter is accompanied by a gauge for the vector potential, allowing access to the highest resolution of its corresponding magnetic field. Since both the order parameter and hopping amplitudes are periodic at lattice sites, we applied Bloch's theorem in order to perform an exact diagonalization of an infinite p-wave superconductor in various vortex states. Thus, we accessed the dispersion of quasi-particle states and studied the formation of Caroli-de Gennes-Matricon states and sub-gap bands induced by tunneling between vortices.

In addition, based on our field theory we generalized the Streda formula to the case of p-wave chiral superconductors. Then, we used the Streda formula to calculate the anomalous charge response, c_{xy} at zero magnetic field and zero vorticity. Due to the periodic boundary conditions the superconductor was probed by the minimal magnetic flux,$\Phi = \pm h/e$ and a compensating vortex pair (placed arbitrarily within the superconductor). In the topological phase ($0 < \mu < 8$) we found that the results of the Streda formula differ from those predicted by the field theory. The discrepancy was traced to the accumulated charge inside the vortices core. Moreover, our study revealed that the formation of CdGM states plays an important role in the charge accumulation process.

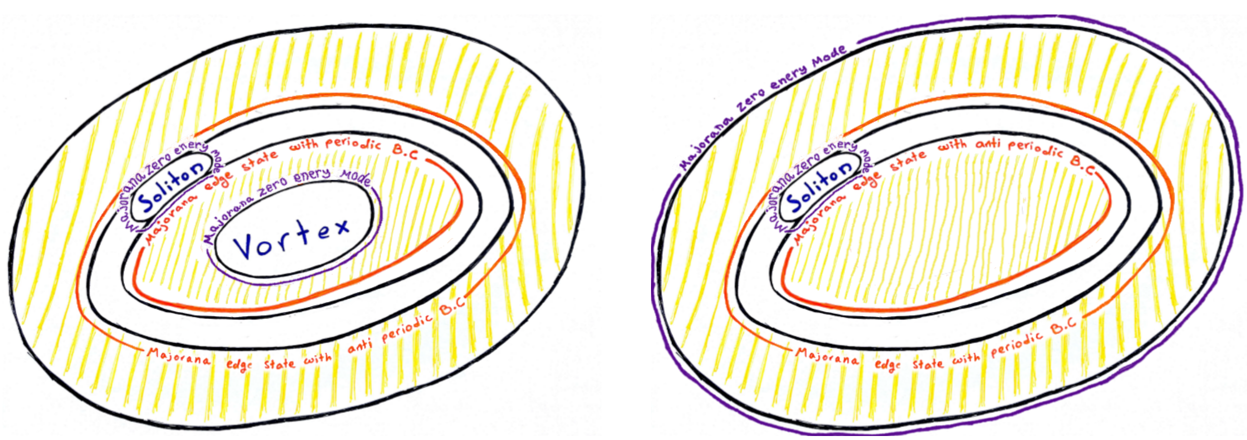

Figure 3.1: **Topological annular Josephson junction.** The boundary conditions of the Majorana edge states depend on the parity of the vortices in the inner plate of the junction.

An accurate estimation of c_{xy} is necessary for determining whether the measurements of the Kerr angle in Sr_2RuO_4 provide evidence for triplet, odd-parity pairing, and chiral order. However, a more realistic lattice model should be used to account for the superconductor multibands if a quantitative comparison is to be made.

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Appendices

Appendix A

The effective action of a two-dimensional chiral p-wave superconductor with single-quantum vortices

1 Introduction

In a p-wave superconductor, the quasi-particles which exhibit non-Abelian statistics are flux $h/2e$ vortices [10, 1]. We present the effective action of such a superconductor [2, 11] and show that when calculations are performed in a certain gauge, they do produce a non-trivial Chern-Simons-type (CS) term.

The Bogoliubov–de Gennes Hamiltonian for a spinless p-wave superconductor is [13, 12]

$$\mathcal{H} = \int_{\mathbb{R}} dt \int_{\mathbb{R}^2} d\boldsymbol{x}\, \psi^{\dagger}_{\boldsymbol{x},t} \left(\frac{(\boldsymbol{p} - \mathbf{A})^2}{2m} - \mu - A_0 \right) \psi_{\boldsymbol{x},t} + \frac{1}{2} \left[\psi_{\boldsymbol{x},t} \left\{ \bar{\Delta}, \boldsymbol{p}_x + i\boldsymbol{p}_y \right\} \psi_{\boldsymbol{x},t} + \text{h.c.} \right] , \tag{1}$$

where $\mathbf{p} = -i\boldsymbol{\nabla}$ is the momentum, ψ, $\psi^{\dagger}$ are the electron field operators, Δ is the order parameter and may depend on space and time, $\mathbf{A}(\mathbf{r})$ is the electromagnetic vector potential. Here and in the following we take $e = \hbar = 1$. The action functional corresponding the Hamiltonian eq. (1) is

$$\mathcal{S}\left(\bar{\phi}_{\boldsymbol{x},t},\phi_{\boldsymbol{x},t}\right)=\int_{-\infty}^{\infty}dt\int_{\mathbf{R}^2}d\boldsymbol{x}\left[\bar{\phi}_{\boldsymbol{x},t}\left(i\partial_t\right)\phi_{\boldsymbol{x},t}-\mathcal{H}\left(\bar{\phi}_{\boldsymbol{x},t},\phi_{\boldsymbol{x},t}\right)\right] \tag{2}$$

where the fermion operators appearing in Eq. (1), $\psi^{\dagger}_{\boldsymbol{x},t}$ and $\psi_{\boldsymbol{x},t}$ were replaced by Grassmann fields, which we denote by $\bar{\phi}_{\boldsymbol{x},t}$ and $\phi_{\boldsymbol{x},t}$, respectively. The partition function of the system is given by the sum over all possible Grassmann field configurations, weighted by the action functional of the fields -

$$\mathcal{Z}=\int\mathcal{D}(\bar{\eta}_{\boldsymbol{x},t},\eta_{\boldsymbol{x},t})e^{i\mathcal{S}(\bar{\eta}_{x,t},\eta_{x,t})} \tag{3}$$

The action is quadratic in the Grassmann fields, and assisted by the Nambu notation the partition function can be integrated out straightforwardly. We use Nambu notation

$$\eta_{\boldsymbol{x},t}=\begin{pmatrix}\phi_{\boldsymbol{x},t}\\ \bar{\phi}_{\boldsymbol{x},t}\end{pmatrix}\text{ and }\bar{\eta}_{\boldsymbol{x},t}=\begin{pmatrix}\bar{\phi}_{\boldsymbol{x},t},\phi_{\boldsymbol{x},t}\end{pmatrix}. \tag{4}$$

Writing the action in terms of Nambu spinors gives

$$\mathcal{S}\left(\bar{\eta}_{\boldsymbol{x},t},\eta_{\boldsymbol{x},t}\right)=\frac{1}{2}\int_{-\infty}^{\infty}dt\int_{\mathbf{R}^2}d\boldsymbol{x}\left[\bar{\eta}_{\boldsymbol{x},t}\mathcal{G}^{-1}\eta_{\boldsymbol{x},t}\right],\text{ where }\mathcal{G}^{-1}=i\partial_t-\mathcal{H} \tag{5}$$

Explicitly, The inverse Green matrix in the presence of electromagnetic fields is

$$\mathcal{G}^{-1}=\begin{pmatrix}i\partial_t+A_0-\frac{1}{2m}(-\boldsymbol{p}+\mathbf{A})^2+\mu & -\left\{\Delta,\boldsymbol{p}_x-i\boldsymbol{p}_y\right\}\\ -\left\{\overline{\Delta},\boldsymbol{p}_x+i\boldsymbol{p}_y\right\} & i\partial_t-A_0+\frac{1}{2m}(\boldsymbol{p}+\mathbf{A})^2-\mu\end{pmatrix},\Delta=\frac{\Delta_0}{2}e^{i\theta(x,t)}. \tag{6}$$

and in terms of the Pauli matrices, The inverse Green matrix with electromagnetic fields is

$$\mathcal{G}^{-1}=i\partial_t-\boldsymbol{\tau}_3\left(\frac{(\boldsymbol{p}-\boldsymbol{\tau}_3\mathbf{A})^2}{2m}-\mu-A_0\right)-\boldsymbol{\tau}_1\{\Delta,\boldsymbol{p}_x\}-\boldsymbol{\tau}_2\{\Delta,\boldsymbol{p}_y\} \tag{7}$$

where $\boldsymbol{\tau}_i$ are the Pauli matrices and the order parameter is $\Delta=\frac{\Delta_0}{2}e^{i\boldsymbol{\tau}_3\theta(\boldsymbol{x},t)}$.

The functional integration over a Gaussian of real Grassmann fields is

$$\mathcal{Z}=\int\mathcal{D}(\bar{\eta}_{\boldsymbol{x},t},\eta_{\boldsymbol{x},t})e^{i\mathcal{S}(\bar{\eta}_{x,t},\eta_{x,t})}=\prod_{\boldsymbol{x},t}\text{Pf}\left(\mathcal{G}^{-1}_{\boldsymbol{x},t}\right)=\exp\left[\frac{1}{2}\text{Tr}\log\left(\mathcal{G}^{-1}_{\boldsymbol{x},t}\right)\right], \tag{8}$$

where TrA stands for $\sum_{\boldsymbol{x},t}\langle\boldsymbol{x},t|\text{tr}A|\boldsymbol{x},t\rangle$ and tr is the trace over the 2×2 Nambu space [6]. The full derivation of the integral's solution is given in Appendix A.

2 The gradient expansion of the effective action for a 2D p-wave superconductor

The partition function is invariant under a similarity transformation of the inverse Green function that keeps the modulus of the Jacobian unity. Any transformation should keep the particle-hole symmetry, which follows from the fact that $\{\Xi, \mathcal{H}\} = 0$ where $\Xi = \boldsymbol{\tau}_1 K$ with K being complex conjugation operator and $\boldsymbol{\tau}_1$ being Pauli matrix in Nambu space. In addition, the transformation is required to be single-valued in order to avoid the need to introduce branch cuts for the Grassmann fields. Also, we would like that the transformation of $\mathcal{G}^{-1}$ produce only terms with powers of the phase function $\theta(\boldsymbol{x}, t)$, in order simplify the gradient expansion of the effective action. Thus, the transformation, $\mathbf{U}$, would have the following properties:

1. To maintain particle-hole symmetry, the right spinor is the transpose conjugate of the left spinor,

$$\mathbf{U} \cdot \eta_{\mathbf{x},t} = \left[\bar{\eta}_{\mathbf{x},t} \cdot \mathbf{U}^{-1}\right]^{\dagger}.$$

2. For particle-hole symmetry, the first element of the spinor is the conjugate of the second element,

$$\bar{\Psi} = (\mathbf{U} \cdot \eta_{\mathbf{x},t})^T \boldsymbol{\tau}_1$$

3. A unity modulus Jacobian implies that $\mid \det(\mathbf{U}) \mid = 1$, so $\mathbf{U}$ is unitary matrix which have the general form

$$\mathbf{U} = e^{i\alpha + \beta\boldsymbol{\tau}_1 + \gamma\boldsymbol{\tau}_2 + \delta\boldsymbol{\tau}_3}$$

4. The transformation should eliminate from the off-diagonal elements of $\mathcal{G}$ the dependence on the order parameter field, $\theta(\tau, \boldsymbol{x})$, from the phase function.

5. The transformation should be single-valued.

The following transformation, which is built from a product of a discrete and continues transformations, fulfills the requirements above:

$$\mathbf{U} = \lambda e^{-i\boldsymbol{\tau}_3\theta/2} \tag{9}$$

The first part is $\lambda = e^{i\gamma(t,\boldsymbol{x})}$, where γ is function that depends, in general, on space and time and gives 0 or π. So the spinors transform by the discrete transformation, $\phi_{\boldsymbol{x},t} \longrightarrow -\phi_{\boldsymbol{x},t}$. The second part is $\exp\left[-i\boldsymbol{\tau}_3\theta/2\right]$, where it should be effectively evaluated as a power series for the exponential function, with ordinary powers replaced by matrix powers. Under this transformation the spinor transforms by a continuous transformation, $\phi_{\boldsymbol{x},t} \longrightarrow \phi_{\boldsymbol{x},t} e^{-i\theta(t,\boldsymbol{x})/2}$.

We renotate the Green function after the transformation as:

$$\mathcal{G}^{-1} = \begin{pmatrix} i\partial_t + a_0 - b_0 - \frac{1}{2m}\left(-\boldsymbol{p} + \boldsymbol{a} - \boldsymbol{b}\right)^2 + \mu & -\Delta\left(\boldsymbol{p}_x - i\boldsymbol{p}_y\right) - \Delta\left(\boldsymbol{b}_x - i\boldsymbol{b}_y\right) \\ -\Delta\left(\boldsymbol{p}_x + i\boldsymbol{p}_y\right) - \Delta\left(\boldsymbol{b}_x + i\boldsymbol{b}_y\right) & i\partial_t - a_0 - b_0 + \frac{1}{2m}\left(\boldsymbol{p} + \boldsymbol{a} + \boldsymbol{b}\right)^2 - \mu \end{pmatrix} \tag{10}$$

where $\boldsymbol{a} = \mathbf{A} - \partial_{\boldsymbol{x}}\theta$, $a_0 = A_0 - \partial_t\theta$, $\boldsymbol{b} = \partial_{\boldsymbol{x}}\gamma$, $b_0 = \partial_t\gamma$ and $\Delta_0 = 2|\Delta|$ (from hereby we omit the zero subscript). The derivation of the transformed Green's matrix can be found in Appendix G.

The similarity transformation is actually a gauge transformation which leaves the Hamiltonain invariant but alter the auxiliary fields $(\Delta, \overline{\Delta})$ as $\Delta \longrightarrow e^{i\theta}\Delta$.

Rewriting the Green matrix in terms of Pauli matrices gives -

$$\mathcal{G}^{-1} = i\partial_t\boldsymbol{\tau}_0 + (a_0 - b_0\boldsymbol{\tau}_3)\boldsymbol{\tau}_3 - \frac{1}{2m}\left(-\boldsymbol{p}\,\boldsymbol{\tau}_3 + \boldsymbol{a} - \boldsymbol{b}\,\boldsymbol{\tau}_3\right)^2\boldsymbol{\tau}_3 + \mu\,\boldsymbol{\tau}_3 + \Delta_0\left(\boldsymbol{p}_x + b_x\right)\boldsymbol{\tau}_1 + \Delta_0\left(\boldsymbol{p}_y + b_y\right)\boldsymbol{\tau}_2$$

Now we divide the expression into the vacuum inverse Green matrix plus corrections that depend on the fields -

$$\begin{aligned}\mathcal{G}^{-1} = \; & \underbrace{i\boldsymbol{\tau}_0\partial_t - \boldsymbol{\tau}_3\left(\frac{\boldsymbol{p}_x{}^2 + \boldsymbol{p}_y{}^2}{2m} - \mu\right) - \Delta\left(\boldsymbol{p}_x\,\boldsymbol{\tau}_1 + \boldsymbol{p}_y\,\boldsymbol{\tau}_2\right)}_{\mathcal{G}_0^{-1}} - \underbrace{\left(-a_0\,\boldsymbol{\tau}_3 - \frac{\{\boldsymbol{p}_x, a_x\} + \{\boldsymbol{p}_y, a_y\}}{2m}\boldsymbol{\tau}_0\right)}_{\chi_1} \\ & - \underbrace{\frac{\{\boldsymbol{p}_x, b_x\} + \{\boldsymbol{p}_y, b_y\}}{2m}\boldsymbol{\tau}_3}_{\chi_2} - \underbrace{\Delta\left(b_x\,\boldsymbol{\tau}_1 + b_y\,\boldsymbol{\tau}_2\right)}_{\chi_3} - \underbrace{b_0\,\boldsymbol{\tau}_0}_{\chi_4} - \underbrace{\frac{a_x^2 + a_y^2}{2m}\boldsymbol{\tau}_3}_{\chi_5} - \underbrace{\frac{-a_x b_x - a_y b_y}{m}\boldsymbol{\tau}_0}_{\chi_6} - \underbrace{\frac{b_x^2 + b_y^2}{2m}\boldsymbol{\tau}_3}_{\chi_7}\end{aligned} \tag{11}$$

This allows us to write the effective action as -

$$\begin{aligned}-iS\left(\boldsymbol{a}, a_0, \boldsymbol{b}, b_0\right) = \; & -\frac{1}{2}\mathrm{Tr}\log\left(\mathcal{G}_0^{-1} - \chi\right) = -\frac{1}{2}\mathrm{Tr}\log(\mathcal{G}_0^{-1}) - \frac{1}{2}\mathrm{Tr}\log\left(\mathbf{1} - \mathcal{G}_0\chi\right) \approx \\ & \mathrm{Const} + \frac{1}{2}\mathrm{Tr}\left[\frac{1}{2}\mathcal{G}_0\cdot\Xi\cdot\mathcal{G}_0\cdot\Xi + \mathcal{G}_0\cdot\chi\right]\end{aligned} \tag{12}$$

The Green's matrix is

$$\mathcal{G}_0 = \frac{-i\boldsymbol{\tau}_0\partial_t - \boldsymbol{g}_{\boldsymbol{p}}\cdot\boldsymbol{\tau}}{\partial_t^2 + \boldsymbol{g}_{\boldsymbol{p}}^2}, \tag{13}$$

where $\boldsymbol{g}_{\boldsymbol{p}} = (\Delta p_x, \Delta p_y, \xi_{\boldsymbol{p}})$, $\xi_{\boldsymbol{p}} = \frac{\boldsymbol{p}^2}{2m} - \mu$, $\boldsymbol{\tau} = (\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}_3)$, $\chi \equiv \sum_{i=1}^{7}\chi_i$ and $\Xi = \sum_{i=1}^{4}\chi_i$.

To derive the Green's matrix we used the fact that every 2D matrix of the form $\mathbf{A} = ia\boldsymbol{\tau}_0 + b\boldsymbol{\tau}_1 + c\boldsymbol{\tau}_2 + d\boldsymbol{\tau}_3$ has an inverse matrix $\mathbf{A}^{-1} = \frac{-ia\boldsymbol{\tau}_0 + b\boldsymbol{\tau}_1 + c\boldsymbol{\tau}_2 + d\boldsymbol{\tau}_3}{a^2 + b^2 + c^2 + d^2}$.

This expansion into a series is called "the gradient expansion" since we assume that the electromagnetic fields and the gradients of the order parameter are small [8, 11, 2].

3 Evaluating the first order of the gradient expansion

We are interested in evaluating the trace of single current terms $\mathcal{G}_0\chi_i$, so we start by examining the trace for the combination $\mathcal{G}_0(\chi_1+\chi_6)$ in order to study how it transforms to momentum space [1] -

$$\begin{aligned}
&\tfrac{1}{2}\int \tfrac{dx}{dt}\langle \boldsymbol{x},t\left|\mathcal{G}_0\left(\chi_1+\chi_4\right)\right|\boldsymbol{x},t\rangle = \tfrac{1}{2}\int \tfrac{dx}{dt}\langle \boldsymbol{x},t\left|\mathcal{G}_0\left(-\boldsymbol{\tau}_3 a_0-\boldsymbol{\tau}_0\tfrac{\{\boldsymbol{p},\boldsymbol{a}\}}{2m}-\boldsymbol{\tau}_0\tfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{m}\right)\right|\boldsymbol{x},t\rangle = \\
&\tfrac{1}{2}\int \tfrac{d\boldsymbol{x},dt}{d\boldsymbol{k},d\omega}\langle \boldsymbol{x},t\left|\mathcal{G}_0\right|\boldsymbol{k},\omega\rangle\langle \boldsymbol{k},\omega\left|\left(-\boldsymbol{\tau}_3 a_0-\boldsymbol{\tau}_0\tfrac{1}{2m}\left(2\boldsymbol{a}\cdot\boldsymbol{p}-i\partial_{\boldsymbol{x}}\boldsymbol{a}\right)-\boldsymbol{\tau}_0\tfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{m}\right)\right|\boldsymbol{x},t\rangle = \\
&\tfrac{1}{2}\tfrac{1}{(2\pi)^3}\int \tfrac{d\boldsymbol{x},dt}{d\boldsymbol{k},d\omega}\mathcal{G}_0(\boldsymbol{k},\omega)\left(-\boldsymbol{\tau}_3 a_0-\boldsymbol{\tau}_0\tfrac{1}{2m}\left(2\boldsymbol{k}\cdot\boldsymbol{a}-i\partial_{\boldsymbol{x}}\boldsymbol{a}\right)-\tfrac{a_x b_x+a_y b_y}{m}\boldsymbol{\tau}_0\right) = \\
&\tfrac{1}{2}\tfrac{1}{(2\pi)^3}\int \tfrac{d\boldsymbol{k},d\omega,}{d\boldsymbol{q},df}\mathcal{G}_0\left(\boldsymbol{k},\omega\right)\left(\left(-\boldsymbol{\tau}_3 a_{0+}-\boldsymbol{\tau}_0\tfrac{\boldsymbol{k}\cdot\boldsymbol{a}_+}{m}\right)\delta_{\boldsymbol{q},0}\delta_{\mathrm{f},0}-\boldsymbol{\tau}_0\tfrac{\boldsymbol{a}_+\cdot\boldsymbol{b}_-}{m}\right)
\end{aligned} \tag{14}$$

where

$$\mathcal{G}(\boldsymbol{k},\omega)=\frac{-\boldsymbol{\tau}_0\omega-\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau}}{-\omega^2+\boldsymbol{g}_{\boldsymbol{k}}^2},\quad g_{\boldsymbol{k}}^2=\xi_{\boldsymbol{k}}^2+\Delta^2\boldsymbol{k}^2,\quad \boldsymbol{g}_{\boldsymbol{k}}=(\Delta k_x,\Delta k_y,\xi_{\boldsymbol{k}}),\ \ \xi_{\boldsymbol{k}}=\frac{\boldsymbol{k}^2}{2m}-\mu\ \text{ and }\ \int\tfrac{dx}{dt}\equiv\int_{\mathbf{R}^2}d\boldsymbol{x}\int_{-\infty}^{\infty}dt$$

and we donate the transformed fields as $\boldsymbol{a}_\pm=\boldsymbol{a}(\pm\boldsymbol{q},\pm f)$ and $\boldsymbol{b}_\pm=\boldsymbol{b}(\pm\boldsymbol{q},\pm f)$. Also should be mentioned that in the last step in Eq. (14) we used Fourier transform identities which are derived in Appendix B.

Thus, when evaluating the trace of terms in the effective action with the form $\mathrm{tr}\left(\int\tfrac{dx}{dt}\langle\boldsymbol{x},t\left|\mathcal{G}_0\chi_i\right|\boldsymbol{x},t\rangle\right)$, the trace can always be transformed to momentum-frequency space, $\mathrm{tr}\left(\int_{\boldsymbol{q},f}^{\boldsymbol{k},\omega}\mathcal{G}_0\left(\boldsymbol{k},\omega_n\right)\chi_i(\boldsymbol{q},f,\boldsymbol{k})\right)$. Where currents that are proportional to $\{\boldsymbol{p},\boldsymbol{a}(\boldsymbol{x},t)\}$, $a_0(\boldsymbol{x},t)$ and $\boldsymbol{a}\cdot\boldsymbol{b}$ become proportional to $2\boldsymbol{k}\cdot\boldsymbol{a}_+\delta_{\boldsymbol{q},0}\delta_{f,0}$, $a_{0+}\delta_{\boldsymbol{q},0}\delta_{f,0}$ and $\boldsymbol{a}_+\cdot\boldsymbol{b}_-$,respectively.

Applying these rules to transform all the first order corrections in the action into the momentum-frequency space, yields:

$$\begin{aligned}
&\frac{1}{2}\int\tfrac{dx}{dt}\,\mathrm{tr}\langle\boldsymbol{x},t\left|\mathcal{G}_0\chi\right|\boldsymbol{x},t\rangle=\frac{1}{2}\frac{1}{(2\pi)^3}\int\tfrac{d\boldsymbol{k},d\omega}{d\boldsymbol{q},df}\mathrm{tr}\left(\mathcal{G}_0\left(\boldsymbol{k},\omega\right)\cdot\boldsymbol{\tau}_3\left(-a_{0+}\delta_{\boldsymbol{q},0}\delta_{\mathrm{f},0}+\frac{\boldsymbol{a}_+\cdot\boldsymbol{a}_-}{2m}+\frac{\boldsymbol{b}_+\cdot\boldsymbol{b}_-}{2m}\right)\right)= \\
&\int\tfrac{d\boldsymbol{q}}{df}\,in\left(-a_{0+}\delta_{\boldsymbol{q},0}\delta_{\mathrm{f},0}+\frac{\boldsymbol{a}_+\cdot\boldsymbol{a}_-}{2m}+\frac{\boldsymbol{b}_+\cdot\boldsymbol{b}_-}{2m}\right)=in\int\tfrac{d\boldsymbol{x}}{dt}\left(-a_0(\boldsymbol{x},t)+\frac{\boldsymbol{a}^2(\boldsymbol{x},t)}{2m}+\frac{\boldsymbol{b}^2(\boldsymbol{x},t)}{2m}\right)
\end{aligned} \tag{15}$$

The corrections χ_6 and the second term in χ_1 do not contribute to the first order of the gradient expansion since the trace and the integration over frequency, ω eliminated these terms. The corrections χ_2,χ_3 and χ_4 do not contribute either, because, although they have the same form as χ_1, the fields are derivatives (of γ). Thus, the Fourier transform

[1]The Fourier Transform convention that we follow is $\langle\boldsymbol{x},t|\boldsymbol{k},\omega\rangle=\frac{\exp(i\boldsymbol{k}\cdot\boldsymbol{x})}{2\pi}\frac{\exp(-i\omega t)}{\sqrt{2\pi}}$

over these terms is zero.

All that's left is to evaluate the sum over $\boldsymbol{k}$ and ω, meaning computing $in \equiv \frac{1}{2}\mathrm{tr}\left(\frac{1}{(2\pi)^3}\int \frac{d\boldsymbol{k}}{d\omega}\,\mathcal{G}_0(\boldsymbol{k},\omega)\boldsymbol{\tau}_3 e^{-i\tau_3\eta}\right)$. The only elements which we need in order to preform the trace in the expression are the diagonal ones. The integration over the momentum of the first (second) element is done by counting only poles in the upper (bottom) half-plane. Thus, the elements were multiplied by an exponents which, due to convergence issues, forces us to choose the right integration contour without altering the result. The sum over poles in the upper (bottom) half-plane may be interpreted as summing over particles (holes).

The expression for the particles and holes can be written as

$$in_{\pm} = \frac{i}{8\pi^2}\int_{\mathbf{R}^2} d\boldsymbol{k}\; n(\boldsymbol{k})_{\pm} = \frac{1}{2}\frac{1}{(2\pi)^3}\int_{\mathbf{R}^2} d\boldsymbol{k}\int_{-\infty}^{\infty} d\omega\;\frac{-\omega \mp \xi_{\boldsymbol{k}}}{-\omega^2 + \boldsymbol{g}_{\boldsymbol{k}}^2 - i\eta}e^{\pm i\eta\omega} \tag{16}$$

Where $\xi_{\boldsymbol{k}} \equiv \frac{\boldsymbol{k}^2}{2m} - \mu$, $\boldsymbol{g}_{\boldsymbol{k}}^2 \equiv \xi^2 + \Delta^2\boldsymbol{k}^2$ and n should be understood as density of particles.

Starting from summing over ω yields

$$\begin{aligned} n(\boldsymbol{k})_{+} &= \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{-\omega - \xi}{-\omega^2 + \boldsymbol{g}^2 - i\eta}e^{i\eta\omega}d\omega = \mathrm{Res}\left(\frac{-\omega-\xi}{(\lambda+\omega)(\lambda-\omega)}e^{i\eta\omega}, -\lambda\right) = \\ &\lim_{\omega\to-\lambda}(\omega+\lambda)\frac{-\omega-\xi}{(\lambda+\omega)(\lambda-\omega)}e^{i\eta\omega} = \frac{\lambda-\xi}{2\lambda}e^{-i\eta\lambda} \end{aligned} \tag{17}$$

and

$$\begin{aligned} n(\boldsymbol{k})_{-} &= \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{-\omega + \xi}{-\omega^2 + \boldsymbol{g}^2 - i\eta}e^{-i\eta\omega}d\omega = \mathrm{Res}\left(\frac{-\omega+\xi}{(\lambda+\omega)(\lambda-\omega)}e^{-i\eta\omega}, \lambda\right) = \\ &-\lim_{\omega\to\lambda}(\omega-\lambda)\frac{-\omega+\xi}{(\lambda+\omega)(\lambda-\omega)}e^{-i\eta\omega} = -\frac{\lambda-\xi}{2\lambda}e^{i\eta\lambda} \end{aligned} \tag{18}$$

where $\lambda \equiv \sqrt{\boldsymbol{g}_{\boldsymbol{k}}^2 - i\eta}$.

We can always write $\lambda^2 = Ce^{i\phi}$ with $C = (g_{\boldsymbol{k}}^4 + \eta^2)^{1/2}$ and $\phi = \arctan(-\eta/g_{\boldsymbol{k}}^2)$. Since $\Im\mathrm{m}(\lambda^2) < 0$, the phase ϕ must be in the region $-\pi \le \phi \le 0$. This implies that $\lambda = \sqrt{C}e^{i\phi/2}$ lies in the forth quarter (and $-\lambda$ in the second quarter).

Taking the trace and integrating over momentum gives -

$$n = \lim_{\eta\to 0} n_+ - n_- = \frac{1}{2}\frac{1}{(2\pi)^2}\int_{\mathbf{R}^2} d\boldsymbol{k}\left(1-\frac{\xi_{\boldsymbol{k}}}{\sqrt{\xi_{\boldsymbol{k}}^2+2m\Delta^2(\xi_{\boldsymbol{k}}+\mu)}}\right) = \frac{m}{4\pi}\int_{-\mu}^{+\Lambda} d\xi\left(1-\frac{\xi}{\sqrt{\xi^2+2m\Delta^2(\xi+\mu)}}\right) =$$
$$\frac{m}{4\pi}\left(\Lambda+2\mu-\sqrt{\Lambda^2+2m\Delta^2(\Lambda+\mu)}+m\Delta^2\log\left[1+\frac{\Lambda+\sqrt{\Lambda^2+2m\Delta^2(\Lambda+\mu)}}{m\Delta^2}\right]\right) \tag{19}$$

where Λ is some cut-off that depends on the nature of the system. It will become useful to introduce the equilibrium electron density as a function of the momentum cut-off, Λ_k.

$$n = \frac{1}{4\pi}\int_0^{\Lambda_k} dk k\left(1-\frac{\xi_k}{\sqrt{\xi_k^2+2m\Delta^2(\xi_k+\mu)}}\right) \tag{20}$$

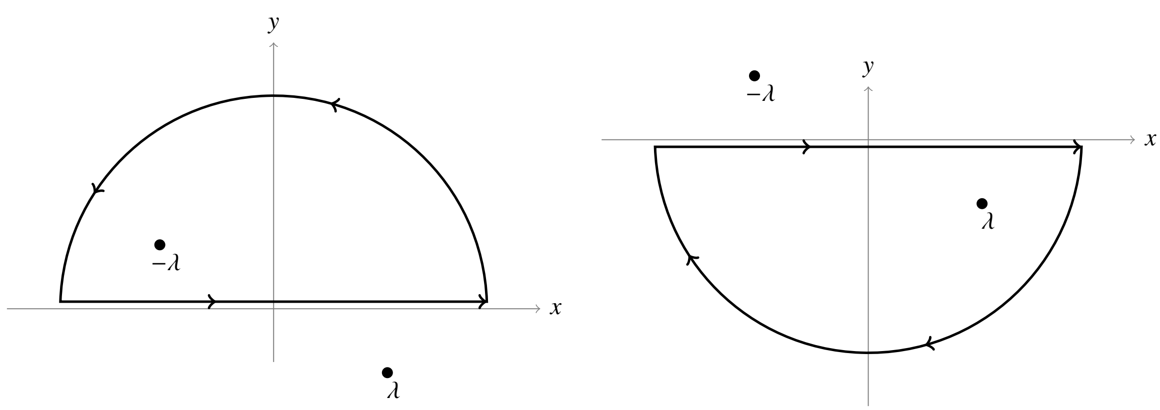

Figure 1: The contours of integration to be used in zero temperature calculations for particles (left) and hole (right).

4 The second order of the gradient expansion

We are interested in evaluating the trace of the second order of the gradient expansion. We start by examining the trace for the combination $\mathcal{G}_0\chi_1\mathcal{G}\chi_1$ in order to study how they transform to frequency-momentum space -

$$\tfrac{1}{4}\int_{dt_0}^{d\boldsymbol{x}_0}\mathrm{tr}\left(\langle\mathbf{x_0},t_0|\mathcal{G}_0\chi_1\mathcal{G}_0\chi_1|\chi_0,t_0\rangle\right) =$$

$$\tfrac{1}{4}\int_{d\boldsymbol{k},d\omega,d\boldsymbol{q},df}^{d\boldsymbol{x}_0,dt_0,d\boldsymbol{x}_1,dt_1}\mathrm{tr}\left(\langle\mathbf{x_0},t_0|\mathcal{G}_0|\boldsymbol{k}_0,\omega_0\rangle\langle\boldsymbol{k}_0,\omega_0|\chi_1|\mathbf{x_1},t_1\rangle\langle\mathbf{x_1},t_1|\mathcal{G}_0|\boldsymbol{k}_1,\omega_1\rangle\langle\boldsymbol{k}_1,\omega_1|\chi_1|\mathbf{x_0},t_0\rangle\right) = \tag{21}$$

$$\tfrac{1}{4}\int_{d\boldsymbol{k},d\omega,d\boldsymbol{q},df}^{d\boldsymbol{x}_0,dt_0,d\boldsymbol{x}_1,dt_1}\mathrm{tr}\left(\tfrac{e^{+i(\mathbf{x_0}-\mathbf{x_1})(\boldsymbol{k}_0-\boldsymbol{k}_1)}e^{+i(t_0-t_1)(\omega_1-\omega_0)}}{(2\pi)^6}\mathcal{G}_0(\boldsymbol{k}_0,\omega_0)\chi_1(\mathbf{x_1},t_1,\boldsymbol{k}_0)\mathcal{G}_0(\boldsymbol{k}_1,\omega_1)\chi_1(\mathbf{x_0},t_0,\boldsymbol{k}_1)\right)$$

where $\mathcal{G}_0(\boldsymbol{k},\omega) = \dfrac{-\omega\boldsymbol{\tau}_0 - \boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau}}{-\omega^2 + \boldsymbol{g}_{\boldsymbol{k}}^2}$ and $\chi_{\mathbf{1}}(\boldsymbol{x},t,\boldsymbol{k}) \;=\; -\boldsymbol{\tau}_0 a_0(\boldsymbol{x},t) - \boldsymbol{\tau}_0\dfrac{1}{2m}\left(2\boldsymbol{k}\cdot\boldsymbol{a}(\boldsymbol{x},t) + i\partial_{\mathbf{x_1}}\boldsymbol{a}(\boldsymbol{x},t)\right)$

Continuing by integrating over the spatiotemporal coordinates yields:

$$\frac{1}{4}\mathrm{tr}\left(\frac{1}{(2\pi)^3}\int_{d\boldsymbol{k}_0,d\omega_0}^{d\boldsymbol{k}_1,d\omega_1}\mathcal{G}_0\left(\boldsymbol{k}_0,\omega_0\right)\chi_{\mathbf{1}}\left(\boldsymbol{k}_0-\boldsymbol{k}_1,\omega_1-\omega_0,\boldsymbol{k}_0\right)\mathcal{G}_0\left(\boldsymbol{k}_1,\omega_1\right)\chi_{\mathbf{1}}\left(\boldsymbol{k}_1-\boldsymbol{k}_0,\omega_0-\omega_1,\boldsymbol{k}_1\right)\right) \tag{22}$$

where the correction
$$\begin{aligned}\chi_1\left(\boldsymbol{k}_0-\boldsymbol{k}_1,\omega_1-\omega_0,\boldsymbol{k}_0\right) &= -\boldsymbol{\tau}_3 a_0\left(\boldsymbol{k}_0-\boldsymbol{k}_1,\omega_1-\omega_0\right)\\ &-\frac{1}{2m}\boldsymbol{\tau}_0\left(2\boldsymbol{k}_0\cdot\boldsymbol{a}\left(\boldsymbol{k}_0-\boldsymbol{k}_1,\omega_1-\omega_0\right)-\left(\boldsymbol{k}_0-\boldsymbol{k}_1\right)\boldsymbol{a}\left(\boldsymbol{k}_0-\boldsymbol{k}_1,\omega_1-\omega_0\right)\right) = \\ &-\boldsymbol{\tau}_3 a_0\left(\boldsymbol{k}_0-\boldsymbol{k}_1,\omega_1-\omega_0\right)-\frac{1}{2m}\boldsymbol{\tau}_0\left(\mathbf{k_0}+\mathbf{k_1}\right)\boldsymbol{a}\left(\boldsymbol{k}_0-\boldsymbol{k}_1,\omega_1-\omega_0\right)\end{aligned} \tag{23}$$

Next, we change the variables of the sum from $\boldsymbol{k}_0$, $\boldsymbol{k}_1$, ω_0 and ω_1 to $\boldsymbol{k} = \frac{\boldsymbol{k}_0+\boldsymbol{k}_1}{2}$, $\boldsymbol{q} = \boldsymbol{k}_0 - \boldsymbol{k}_1$, $\omega = \frac{\omega_1+\omega_0}{2}$ and $f = \omega_1 - \omega_0$, respectively. After the variables change, the contribution to the action takes the form

$$\frac{1}{4}\mathrm{tr}\left(\frac{1}{(2\pi)^3}\int_{d\boldsymbol{q},df}^{d\boldsymbol{k},d\omega}\mathcal{G}_0\left(\boldsymbol{k}+\frac{\boldsymbol{q}}{2},\omega-\frac{f}{2}\right)\chi_{\mathbf{1}}\left(\boldsymbol{q},f,\boldsymbol{k}\right)\mathcal{G}_0\left(\boldsymbol{k}-\frac{\boldsymbol{q}}{2},\omega+\frac{f}{2}\right)\chi_{\mathbf{1}}\left(-\mathbf{q},-f,\boldsymbol{k}\right)\right) \tag{24}$$

where $\mathcal{G}_0(\omega\mp\dfrac{f}{2},\boldsymbol{k}\pm\dfrac{\boldsymbol{q}}{2}) = \dfrac{-(\omega\mp\frac{f}{2})\boldsymbol{\tau}_0 - \boldsymbol{g}_{\boldsymbol{k}\pm\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau}}{-(\omega-\frac{f}{2})^2 + \boldsymbol{g}^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}}$ and $\chi_{\mathbf{1}}(\boldsymbol{q},f,\boldsymbol{k}) = -\boldsymbol{\tau}_3 a_0(\boldsymbol{q},f) - \dfrac{1}{m}\boldsymbol{\tau}_0\boldsymbol{k}\cdot\boldsymbol{a}(\boldsymbol{q},f).$

Generally speaking, when transforming terms of the form $\mathrm{tr}\int\frac{d\boldsymbol{x}}{dt}\langle\boldsymbol{x},t\left|\mathcal{G}_0\chi_i\mathcal{G}_0\chi_j\right|\boldsymbol{x},t\rangle$ to the momentum-frequency space, the expressions can be simplified by switching to symmetric and anti-symmetric variables by a special unitary transformation. After the transformation the corrections that are proportional $\{\boldsymbol{p},\boldsymbol{a}(\boldsymbol{x},t)\}$ and $a_0(\boldsymbol{x},t)$ become proportional to $2\boldsymbol{k}\cdot\boldsymbol{a}(\boldsymbol{q},f)$ and $a_0(\boldsymbol{q},f)$, respectively. Thus, the second order of the gradient expansion in terms of (2+1) momentum-frequency vectors $q\equiv(\boldsymbol{q},-q_0)$ and $k\equiv(\boldsymbol{k},k_0)$ can be expressed as

$$\int q_0 d\boldsymbol{q}\;\kappa_\mu(\boldsymbol{q},q_0)\pi^{\mu\nu}_{\kappa\lambda}\lambda_\nu(-\boldsymbol{q},-q_0) \tag{25}$$

where $\pi^{\mu\nu}_{\kappa\lambda}(q)$, the correlator[2] between the μ component of the field κ and the ν component of the field λ, is given by

$$\pi^{\mu\nu}_{\kappa\lambda}(q) = \frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\;\mathrm{tr}\left(\mathcal{G}(k+\frac{q}{2})j^{\mu}_{\kappa}(k)\mathcal{G}(k-\frac{q}{2})j^{\nu}_{\lambda}(k)\right), \tag{26}$$

where the Einstein summation rule is applied with $\kappa,\lambda = a,b$, $\mu,\nu = 0,1,2$ and the currents j_a and j_b are

$$j_a = (-\boldsymbol{\tau}_3, -\tfrac{k_x}{m}\boldsymbol{\tau}_0, -\tfrac{k_y}{m}\boldsymbol{\tau}_0),\quad j_b = (\boldsymbol{\tau}_0, \tfrac{k_x}{m}\boldsymbol{\tau}_3 + \Delta\boldsymbol{\tau}_1, \tfrac{k_y}{m}\boldsymbol{\tau}_3 + \Delta\boldsymbol{\tau}_2). \tag{27}$$

As shown in Appendix G, there is no coupling between the 3-vector fields $a = (\boldsymbol{a}, a_0)$ and $b = (\boldsymbol{b}, b_0)$. This statement is based on the antisymmetry with respect to k of correlators integrand,

$$\mathrm{tr}\left(\mathcal{G}_0(k+\frac{q}{2})j_a(k)\mathcal{G}_0(k-\frac{q}{2})j_b(k)\right) = -\mathrm{tr}\left(\mathcal{G}_0(-k+\frac{q}{2})j_a(-k)\mathcal{G}_0(-k-\frac{q}{2})j_b(-k)\right). \tag{28}$$

The calculations can be further simplified by expanding the correlators up to the first order with respect to q. The expansion of the effective action is only up to the second order of the fields a and b so there is no point to keep orders higher than two in q. Explicitly, the expanded correlators are

$$\Pi^{\mu\nu}_{\kappa}(q) = \frac{1}{4}\frac{1}{(2\pi)^3}(1 + q \cdot \nabla_{q'}) \left[\int d^3k \; \mathrm{tr}\left(\mathcal{G}_0(k + \frac{q'}{2}) j^{\mu}_{\kappa}(\boldsymbol{k}) \mathcal{G}_0(k - \frac{q'}{2}) j^{\nu}_{\kappa}(\boldsymbol{k})\right)\right]\Bigg|_{q'=0} . \tag{29}$$

5 Evaluating the zero order in q of the gradient expansion second order

The zero order in q of correlator $\Pi^{(0)\mu\nu}_{\kappa}(q)$ in terms of $\mathcal{G}^{(0)}_{k} \equiv \mathcal{G}_0(k)$ is

$$\Pi^{(0)\mu\nu}_{\kappa}(q) = \frac{1}{4}\frac{1}{(2\pi)^3}\left[\int \mathrm{d}^3k \; \mathrm{tr}\left(\mathcal{G}^{(0)}_{k} j^{\mu}_{\kappa}(\boldsymbol{k}) \mathcal{G}^{(0)}_{k} j^{\nu}_{\kappa}(\boldsymbol{k})\right)\right] \tag{30}$$

where

$$\mathcal{G}^{(0)}_{k} = \frac{-\boldsymbol{\tau}_0 k_0 - \boldsymbol{g}_{\boldsymbol{k}} \cdot \boldsymbol{\tau}}{-k_0^2 + \boldsymbol{g}_{\boldsymbol{k}}^2}, \quad g^2_{\boldsymbol{k}} = \xi^2_{\boldsymbol{k}} + \Delta^2 \boldsymbol{k}^2 \quad \text{and } \boldsymbol{g}_{\boldsymbol{k}} = (\Delta k_x, \Delta k_y, \xi_{\boldsymbol{k}}) \tag{31}$$

and the currents j_a and j_b are

$$\begin{aligned} j_a(\boldsymbol{k}) &= (-\boldsymbol{\tau}_3, -\tfrac{k_1}{m}\boldsymbol{\tau}_0, -\tfrac{k_2}{m}\boldsymbol{\tau}_0) = \left(-\boldsymbol{\tau}_3, -\partial_{k_1} g_3 \boldsymbol{\tau}_0, -\partial_{k_2} g_3 \boldsymbol{\tau}_0\right) \\ j_b(\boldsymbol{k}) &= (\boldsymbol{\tau}_0, \tfrac{k_1}{m}\boldsymbol{\tau}_3 + \Delta\boldsymbol{\tau}_1, \tfrac{k_2}{m}\boldsymbol{\tau}_3 + \Delta\boldsymbol{\tau}_2) = \left(\boldsymbol{\tau}_0, \partial_{k_1}(\boldsymbol{g} \cdot \boldsymbol{\tau}), \partial_{k_2}(\boldsymbol{g} \cdot \boldsymbol{\tau})\right) \end{aligned} \tag{32}$$

5.1 Intermezzo: The integration over k_0

Fortunately, the calculation of the correlators involves only two types of integrals over the frequency, k_0. These integrals are

$$\begin{aligned} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\omega^2}{(-\omega^2 + \lambda^2)^2} d\omega \quad &= \quad \mathrm{Res}\left(\frac{\omega^2}{(\lambda - \omega)^2(\lambda + \omega)^2}, -\lambda\right) = \lim_{\omega \to -\lambda} \frac{\partial}{\partial \omega}\frac{\omega^2}{(\lambda - \omega)^2} = \\ & \lim_{\omega \to -\lambda}\left(\frac{2\omega}{(\lambda - \omega)^2} + \frac{2\omega^2}{(\lambda - \omega)^3}\right) = -\frac{\lambda^2}{4\lambda^3} \end{aligned} \tag{33}$$

[2] In order to evaluate the correlators we must take a trace over various products of the Pauli matrices and the identity matrix. The following identities may ease the calculation:

$$\mathrm{tr}\left(\boldsymbol{\tau}_i \boldsymbol{\tau}_j \boldsymbol{\tau}_k \boldsymbol{\tau}_l\right) = 2\delta_{ij}\delta_{kl} + 2\delta_{il}\delta_{jk} - 2\delta_{ik}\delta_{jl}, \quad \mathrm{tr}\left(\boldsymbol{\tau}_i \boldsymbol{\tau}_j \boldsymbol{\tau}_k\right) = \epsilon_{ijk} 2i \quad \text{and } \mathrm{tr}\left(\boldsymbol{\tau}_i \boldsymbol{\tau}_j\right) = 2\delta_{i,j}.$$

and

$$\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{1}{(-\omega^2+\lambda^2)^2}d\omega \quad = \quad \mathrm{Res}\left(\frac{1}{(\lambda-\omega)^2(\lambda+\omega)^2},-\lambda\right)=\lim_{\omega\to-\lambda}\frac{\partial}{\partial\omega}\frac{1}{(\lambda-\omega)^2}= \tag{34}$$
$$\lim_{\omega\to-\lambda}\frac{2}{(\lambda-\omega)^3}=\frac{1}{4\lambda^3}$$

with $\lambda \equiv \sqrt{\boldsymbol{g}_{\boldsymbol{k}}^2 - i\eta}$. The results of the integrations do not depend in which half-plane the arc contour lies. Thus, without ambiguity we choose to take the arc in upper half-plane

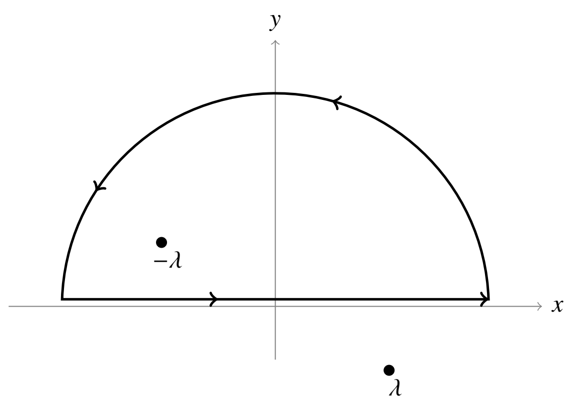

Figure 2: The contour of integration which is used in the evaluation of dual current terms.

5.2 The correlator Π_a

$$\Pi_a^{00}=\frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3}\mathrm{d}^3k\,\mathrm{tr}\left(\mathcal{G}^{(0)}j_a^0\mathcal{G}^{(0)}j_a^0\right)=\frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3}\mathrm{d}^3k\ \mathrm{tr}\left(\frac{(k_0\boldsymbol{\tau}_0+\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_3(k_0\boldsymbol{\tau}_0+\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_3}{(-k_0^2+g^2)^2}\right)=$$
$$\frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3}\mathrm{d}^3k\ \mathrm{tr}\left(\frac{(k_0\boldsymbol{\tau}_0+\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau})(k_0\boldsymbol{\tau}_0+\boldsymbol{g}_{-\boldsymbol{k}}\cdot\boldsymbol{\tau})}{\left(-k_0^2+g^2\right)^2}\right)=\frac{1}{2}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3}\mathrm{d}^3k\ \frac{k_0^2+\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{g}_{-\boldsymbol{k}}}{\left(-k_0^2+g^2\right)^2}=$$
$$\frac{i}{8}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}\mathrm{d}^2k\ \frac{-g^2+\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{g}_{-\boldsymbol{k}}}{g^3}=-\frac{i}{16\pi^2}\iint_{\mathbb{R}^2}\mathrm{d}^2k\ \frac{g_1^2+g_2^2}{g^3}=-\frac{im}{4}\frac{1}{2\pi}\int_{-\mu}^{\infty}d\xi\ \frac{2m\Delta^2(\xi+\mu)}{\xi_{\boldsymbol{k}}^2+2m\Delta^2(\xi+\mu))^{3/2}}=-\frac{im}{4\pi} \tag{35}$$

$$\Pi_a^{11} = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k \,\mathrm{tr}\left(\mathcal{G}^{(0)} j_a^1 \mathcal{G}^{(0)} j_a^1\right) = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k\ \mathrm{tr}\left(\frac{(\partial_{k_1} g_3)^2 (k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})^2}{(-k_0^2+g^2)^2}\right) =$$
$$\left\{\begin{array}{c}\text{Now we use the identity}\\ \mathrm{tr}\left[(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})^2\right] = \\ 2(k_0^2+g^2)\end{array}\right\} = \frac{1}{2}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k\ \frac{(\partial_{k_1} g_3)^2 (k_0^2+g^2)}{(-k_0^2+g^2)^2} = \frac{i\pi}{8\pi}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k\ \frac{(\partial_{k_1} g_3)^2(-g^2+g^2)}{g^3} = 0 \tag{36}$$

$$\Pi_a^{22} = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k \,\mathrm{tr}\left(\mathcal{G}^{(0)} j_a^2 \mathcal{G}^{(0)} j_a^2\right) = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k\ \mathrm{tr}\left(\frac{(\partial_{k_2} g_3)^2 (k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})^2}{(-k_0^2+g^2)^2}\right) =$$
$$\left\{\begin{array}{c}\text{Now we use the identity}\\ \mathrm{tr}\left[(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})^2\right] = \\ 2(k_0^2+g^2)\end{array}\right\} = \frac{1}{2}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k\ \frac{(\partial_{k_2} g_3)^2 (k_0^2+g^2)}{(-k_0^2+g^2)^2} = \frac{i\pi}{8\pi}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k\ \frac{(\partial_{k_2} g_3)^2(-g^2+g^2)}{g^3} = 0 \tag{37}$$

$$\Pi_a^{01} = \Pi_a^{10} = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k \,\mathrm{tr}\left(\mathcal{G}^{(0)} j_a^0 \mathcal{G}^{(0)} j_a^1\right) =$$
$$\frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k\ \mathrm{tr}\left(\frac{(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_3(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})(\partial_{k_1} g_3\boldsymbol{\tau}_0)}{(-k_0^2+g^2)^2}\right) =$$
$$\frac{i}{16}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k\ \mathrm{tr}\left(\frac{[-g^2\boldsymbol{\tau}_3 + (\boldsymbol{g}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_3(\boldsymbol{g}\cdot\boldsymbol{\tau})]\partial_{k_1} g_3}{g^3}\right) = \left\{\begin{array}{c}\text{Now we use the identity}\\ \mathrm{tr}\left[(\boldsymbol{g}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_3(\boldsymbol{g}\cdot\boldsymbol{\tau})\right] = 0\end{array}\right\} = 0 \tag{38}$$

$$\Pi_a^{02} = \Pi_a^{20} = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k \,\mathrm{tr}\left(\mathcal{G}^{(0)} j_a^0 \mathcal{G}^{(0)} j_a^2\right) =$$
$$\frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k\ \mathrm{tr}\left(\frac{(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_3(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})(\partial_{k_2} g_3\boldsymbol{\tau}_0)}{(-k_0^2+g^2)^2}\right) =$$
$$\frac{i}{16}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k\ \mathrm{tr}\left(\frac{[-g^2\boldsymbol{\tau}_3 + (\boldsymbol{g}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_3(\boldsymbol{g}\cdot\boldsymbol{\tau})]\partial_{k_2} g_3}{g^3}\right) = \left\{\begin{array}{c}\text{Now we use the identity}\\ \mathrm{tr}\left[(\boldsymbol{g}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_3(\boldsymbol{g}\cdot\boldsymbol{\tau})\right] = 0\end{array}\right\} = 0 \tag{39}$$

$$\Pi_a^{12} = \Pi_a^{21} = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k \operatorname{tr}\left(\mathcal{G}^{(0)} j_a^1 \mathcal{G}^{(0)} j_a^2\right) = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k \operatorname{tr}\left(\frac{(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})^2(\partial_{k_1}g_3)(\partial_{k_2}g_3)}{(-k_0^2+g^2)^2}\right) =$$

$$\left\{\begin{array}{c}\text{Now we use the identity}\\ \operatorname{tr}\left[(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})^2\right] = \\ 2(k_0^2+g^2)\end{array}\right\} = \frac{1}{2}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k\, \frac{(k_0^2+g^2)(\partial_{k_1}g_3)(\partial_{k_2}g_3)}{(-k_0^2+g^2)^2} =$$

$$\frac{i}{8}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k\, \frac{(-g^2+g^2)(\partial_{k_1}g_3)(\partial_{k_2}g_3)}{g^3} = 0 \quad (40)$$

5.3 The correlator Π_b

$$\Pi_b^{00} = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k \operatorname{tr}\left(\mathcal{G}^{(0)} j_b^0 \mathcal{G}^{(0)} j_b^0\right) = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k \operatorname{tr}\left(\frac{(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_0(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_0}{-k_0^2+g^2}\right) =$$

$$\frac{1}{2}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k \frac{k_0^2+g^2}{\left(-k_0^2+g^2\right)^2} = \frac{i}{8}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k \frac{-g^2+g^2}{g^3} = 0$$

$$(41)$$

$$\Pi_b^{22} = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k \operatorname{tr}\left(\mathcal{G}^{(0)} j_b^2 \mathcal{G}^{(0)} j_b^2\right) = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k \operatorname{tr}\left[\left(\frac{(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})\partial_{k_2}\boldsymbol{g}\cdot\boldsymbol{\tau}}{-k_0^2+g^2}\right)^2\right] =$$

$$= \frac{i}{16}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k \operatorname{tr}\left(\frac{-g^2(\partial_{k_2}\boldsymbol{g}\cdot\boldsymbol{\tau})^2 + (\boldsymbol{g}\cdot\boldsymbol{\tau}\partial_{k_2}\boldsymbol{g}\cdot\boldsymbol{\tau})^2}{g^3}\right) =$$

$$= \frac{i}{16}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k \frac{-2g^2(\partial_{k_2}\boldsymbol{g})^2 + \operatorname{tr}\left[\left(\begin{array}{c}(g_2\partial_{k_2}g_2 + g_3\partial_{k_2}g_3)\boldsymbol{\tau}_0 + i(g_2\partial_{k_2}g_3 - g_3\partial_{k_2}g_2)\boldsymbol{\tau}_1\\ -ig_1\partial_{k_2}g_3\boldsymbol{\tau}_2 + ig_1\partial_{k_2}g_2\boldsymbol{\tau}_3\end{array}\right)^2\right]}{g^3}$$

$$= \left\{\begin{array}{c}\text{Now we use the identity}\\ \operatorname{tr}\left[(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})^2\right]\\ = 2(k_0^2+g^2)\end{array}\right\} = -\frac{i}{4}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k \left[\frac{(g_2\partial_{k_2}g_3 - g_3\partial_{k_2}g_2)^2 + (g_1\partial_{k_2}g_3)^2 + (g_1\partial_{k_2}g_2)^2}{g^3}\right] =$$

$$-\frac{i}{4}\frac{\Delta^2}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k \frac{\left(\frac{k_1^2}{m} - \left(\frac{k^2}{2m} - \mu\right)\right)^2 + \left(\frac{k_1k_2}{m}\right)^2 + \Delta^2 k_2^2}{g^3}$$

$$-\frac{i}{4}\frac{\Delta^2}{(2\pi)^2}\int_0^\infty \mathrm{d}k \int_0^{2\pi} \mathrm{d}\alpha\, k\frac{\left(\frac{k^2\cos^2\alpha}{m} - \left(\frac{k^2}{2m} - \mu\right)\right)^2 + \left(\frac{k}{2m}\right)^2 \sin^2 2\alpha + (\Delta k)^2\cos^2\alpha}{g^3}$$

$$= \lim_{\Lambda_k\to\infty}\left[-\frac{i}{8}\frac{\Delta^2}{2\pi}\int_0^{\Lambda_k} k\,dk\, \frac{\left(\frac{k^4}{2m^2} + \Delta^2k^2 + 2\mu^2\right)}{(\xi_k^2 + \Delta^2k^2)^{3/2}}\right] = -i\left(\frac{1}{4}\partial_m n + \frac{m\Delta^2+\mu}{8\pi}\right) \quad (42)$$

$$\Pi_b^{11} = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k \mathrm{tr}\left(\mathcal{G}^{(0)} j_b^1 \mathcal{G}^{(0)} j_b^1\right) = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k \ \mathrm{tr}\left[\left(\frac{(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})\partial_{k_1}\boldsymbol{g}\cdot\boldsymbol{\tau}}{-k_0^2+g^2}\right)^2\right] =$$

$$= \frac{i}{16}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k \ \mathrm{tr}\left(\frac{-g^2(\partial_{k_1}\boldsymbol{g}\cdot\boldsymbol{\tau})^2 + (\boldsymbol{g}\cdot\boldsymbol{\tau}\partial_{k_1}\boldsymbol{g}\cdot\boldsymbol{\tau})^2}{g^3}\right) =$$

$$= \frac{i}{16}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k \frac{-2g^2(\partial_{k_1}\boldsymbol{g})^2 + \mathrm{tr}\left[\begin{pmatrix} (g_1\partial_{k_1}g_1 + g_3\partial_{k_1}g_3)\boldsymbol{\tau}_0 + ig_2\partial_{k_1}g_3\boldsymbol{\tau}_1 \\ +i(g_3\partial_{k_1}g_1 - g_1\partial_{k_1}g_3)\boldsymbol{\tau}_2 - ig_2\partial_{k_1}g_1\boldsymbol{\tau}_3\end{pmatrix}^2\right]}{g^3}$$

$$= \left\{\begin{array}{c}\text{Now we use the identity} \\ \mathrm{tr}\left[(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})^2\right] \\ = 2(k_0^2+g^2)\end{array}\right\} = -\frac{i}{4}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k\left[\frac{(g_2\partial_{k_1}g_3)^2 + (g_3\partial_{k_1}g_1 - g_1\partial_{k_1}g_3)^2 + (g_2\partial_{k_1}g_1)^2}{g^3}\right] =$$

$$\left\{\begin{array}{c}\text{Switching between the coordinates indices 1 and 2 would not alter the integral} \\ \text{and would result an integrand which is identitical to the one of } \Pi_b^{22}\end{array}\right\}$$

$$= \lim_{\Lambda_k\to\infty}\left[-\frac{i}{8}\frac{\Delta^2}{2\pi}\int_0^{\Lambda_k} kdk \ \frac{\left(\frac{k^4}{2m^2} + \Delta^2k^2 + 2\mu^2\right)}{(\xi_k^2 + \Delta^2k^2)^{3/2}}\right] = -i\left(\frac{1}{4}\partial_m n + \frac{m\Delta^2+\mu}{8\pi}\right) \quad (43)$$

One should notice that Π_b^{11}, Π_b^{22} and the corresponding terms from the first order of the gradient expansion cancels each other as $\Lambda_k \to \infty$.

$$\Pi_b^{12} = \Pi_b^{21} = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k \mathrm{tr}\left(\mathcal{G}^{(0)} j_b^1 \mathcal{G}^{(0)} j_b^2\right) =$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k \ \mathrm{tr}\left[\frac{(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})(\partial_{k_1}\boldsymbol{g}\cdot\boldsymbol{\tau})(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})(\partial_{k_2}\boldsymbol{g}\cdot\boldsymbol{\tau})}{\left(-k_0^2+g^2\right)^2}\right] =$$

$$\frac{i}{16}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k \ \mathrm{tr}\left[\frac{-g^2(\partial_{k_1}\boldsymbol{g}\cdot\boldsymbol{\tau})(\partial_{k_2}\boldsymbol{g}\cdot\boldsymbol{\tau}) + (\boldsymbol{g}\cdot\boldsymbol{\tau})(\partial_{k_1}\boldsymbol{g}\cdot\boldsymbol{\tau})(\boldsymbol{g}\cdot\boldsymbol{\tau})(\partial_{k_2}\boldsymbol{g}\cdot\boldsymbol{\tau})}{g^3}\right] =$$

$$\frac{i}{8}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k \ \frac{1}{g^3}\left[-g^2(\partial_{k_1}g_3)(\partial_{k_2}g_3) + g_1(\partial_{k_1}g_1)g_2(\partial_{k_2}g_2) + g_2(\partial_{k_1}g_1)g_1(\partial_{k_2}g_2) + g_1(\partial_{k_1}g_1)g_3(\partial_{k_2}g_3) + \right.$$

$$g_3(\partial_{k_1}g_1)g_1(\partial_{k_2}g_3) + g_3(\partial_{k_1}g_3)g_2(\partial_{k_2}g_2) + g_2(\partial_{k_1}g_3)g_3(\partial_{k_2}g_2) - g_1(\partial_{k_1}g_3)g_1(\partial_{k_2}g_3)$$

$$\left. -g_2(\partial_{k_1}g_3)g_2(\partial_{k_2}g_3) + g_3(\partial_{k_1}g_3)g_3(\partial_{k_2}g_3)\right] = \left\{\begin{array}{c}\text{All the terms in the integrad are odd} \\ \text{with respect to } k_1 \text{ or } k_2\end{array}\right\} = 0 \quad (44)$$

$$\Pi_b^{01} = \Pi_b^{10} = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k\ \mathrm{tr}\left(\mathcal{G}^{(0)} j_b^0 \mathcal{G}^{(0)} j_b^1\right) =$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k\ \mathrm{tr}\left[\frac{(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_0(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})(\partial_{k_1}\boldsymbol{g}\cdot\boldsymbol{\tau})}{\left(-k_0^2+g^2\right)^2}\right] =$$

$$\frac{i}{16}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k\ \mathrm{tr}\left[\frac{-g^2\partial_{k_1}\boldsymbol{g}\cdot\boldsymbol{\tau} + (\boldsymbol{g}\cdot\boldsymbol{\tau})^2\partial_{k_1}\boldsymbol{g}\cdot\boldsymbol{\tau}}{g^3}\right] = \left\{\ (\boldsymbol{g}\cdot\boldsymbol{\tau})^2 = g^2\boldsymbol{\tau}_0\ \right\} = 0 \quad (45)$$

$$\Pi_b^{02} = \Pi_b^{20} = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k\ \mathrm{tr}\left(\mathcal{G}^{(0)} j_b^0 \mathcal{G}^{(0)} j_b^2\right) =$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\iiint_{\mathbb{R}^3} \mathrm{d}^3k\ \mathrm{tr}\left[\frac{(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_0(k_0\boldsymbol{\tau}_0 + \boldsymbol{g}\cdot\boldsymbol{\tau})(\partial_{k_2}\boldsymbol{g}\cdot\boldsymbol{\tau})}{\left(-k_0^2+g^2\right)^2}\right] =$$

$$\frac{i}{16}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k\ \mathrm{tr}\left[\frac{-g^2\partial_{k_2}\boldsymbol{g}\cdot\boldsymbol{\tau} + (\boldsymbol{g}\cdot\boldsymbol{\tau})^2\partial_{k_2}\boldsymbol{g}\cdot\boldsymbol{\tau}}{g^3}\right] = \left\{\ (\boldsymbol{g}\cdot\boldsymbol{\tau})^2 = g^2\boldsymbol{\tau}_0\ \right\} = 0 \quad (46)$$

6 Evaluating the first order with respect to q of the gradient expansion second order

The first order in q of the Green matrix can be written in following form

$$q\cdot\boldsymbol{\nabla}_{q'}\ \mathcal{G}_0(k+\frac{q'}{2})\Big|_{q'=0} = \frac{\frac{q_0}{2}\boldsymbol{\tau}_0 - \frac{\boldsymbol{k}\cdot\boldsymbol{q}}{2m}\boldsymbol{\tau}_3 - \frac{q_1}{2}\Delta\boldsymbol{\tau}_1 - \frac{q_2}{2}\Delta\boldsymbol{\tau}_2}{-k_0^2+\boldsymbol{g}_{\boldsymbol{k}}^2} - \frac{q_0k_0 + \left(\Delta^2 + \frac{\xi_{\boldsymbol{k}}}{m}\right)\boldsymbol{k}\cdot\boldsymbol{q}}{-k_0^2+\boldsymbol{g}_{\boldsymbol{k}}^2}\mathcal{G}_0(k) \quad (47)$$

The second term in the sum, which we refer as $G^{(1b)}$, do no contribute to the correlator since the function that multiplies $\mathcal{G}_0(k)$ is anti-symmetric with respect to q, .i.e.

$$\mathcal{G}_{k,q}^{(1b)} j_\lambda^\mu \mathcal{G}_k^{(0)} j_\lambda^\nu = \mathcal{G}_k^{(0)} j_\lambda^\mu \mathcal{G}_{k,q}^{(1b)} j_\lambda^\nu = -\mathcal{G}_k^{(0)} j_\lambda^\mu \mathcal{G}_{k,-q}^{(1b)} j_\lambda^\nu. \quad (48)$$

Thus, the zero order in q of correlator $\Pi^{(1)\mu\nu}_{\ \lambda}(q)$ can be written as

$$\Pi^{(1)\mu\nu}_{\ \lambda}(q) \quad = \quad \frac{1}{4}\frac{1}{(2\pi)^3}(q\cdot\boldsymbol{\nabla}_{q'})\left[\iiint d^3k\ \mathrm{tr}\left(\mathcal{G}_0(k+\frac{q'}{2})j_\lambda^\mu(\boldsymbol{k})\mathcal{G}_0(k-\frac{q'}{2})j_\lambda^\nu(\boldsymbol{k})\right)\right]\Big|_{q'=0} = \quad (49)$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\left[\iiint \mathrm{d}^3k\ \mathrm{tr}\left(\mathcal{G}_k^{(0)} j_\lambda^\mu(\boldsymbol{k})\mathcal{G}_{k,-q}^{(1)} j_\lambda^\nu(\boldsymbol{k}) + \mathcal{G}_{k,q}^{(1)} j_\lambda^\mu(\boldsymbol{k})\mathcal{G}_k^{(0)} j_\lambda^\nu(\boldsymbol{k})\right)\right] \quad (50)$$

where

$$\mathcal{G}_k^{(0)} = \frac{-k_0\boldsymbol{\tau}_0 - \boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau}}{-k_0^2+g^2}, \mathcal{G}_{k,q}^{(1)} = \frac{1}{2}\frac{q_0\boldsymbol{\tau}_0 - \boldsymbol{q}\cdot\boldsymbol{\nabla}_{\boldsymbol{k}}(\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau})}{-k_0^2+g^2}, \boldsymbol{g}_{\boldsymbol{k}} = (\Delta k_x, \Delta k_y, \xi_{\boldsymbol{k}}) \quad (51)$$

and the currents j_a and j_b are

$$j_a = -(\boldsymbol{\tau}_3, \partial_{k_1} g_3 \boldsymbol{\tau}_0, \partial_{k_2} g_3 \boldsymbol{\tau}_0), \quad j_b = \big(\boldsymbol{\tau}_0, \partial_{k_1}(\boldsymbol{g}\cdot\boldsymbol{\tau}), \partial_{k_2}(\boldsymbol{g}\cdot\boldsymbol{\tau})\big) \tag{52}$$

6.1 The correlator Π_a

$$\Pi^{(1)01}_{a}(q) = -\Pi^{(1)10}_{a}(q) = \frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3 k \ \mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^0_a \mathcal{G}^{(0)}_k j^1_a + \mathcal{G}^{(0)}_k j^0_a \mathcal{G}^{(1)}_{k,-q} j^1_a\right) =$$

$$-\frac{1}{8}\frac{1}{(2\pi)^3}\iiint \mathrm{d}^3 k \ \mathrm{tr}\left(\frac{(q_0\boldsymbol{\tau}_0 - \boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g_k}\cdot\boldsymbol{\tau}))\boldsymbol{\tau}_3(\boldsymbol{\tau}_0 k_0 + \boldsymbol{g_k}\cdot\boldsymbol{\tau}) - (\boldsymbol{\tau}_0 k_0 + \boldsymbol{g_k}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_3(q_0\boldsymbol{\tau}_0 - \boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g_k}\cdot\boldsymbol{\tau}))}{(-k_0^2 + g^2)^2}\partial_{k_1} g_3\right) =$$

$$\left\{\begin{array}{c}q_0 \text{ is proportional to} \\ (\boldsymbol{g}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_3 - \boldsymbol{\tau}_3(\boldsymbol{g}\cdot\boldsymbol{\tau}) \\ \text{which has zero trace} \\ \text{and we integrate over } k_0.\end{array}\right\} = \frac{i}{32}\frac{1}{(2\pi)^2}\iint \mathrm{d}^2 k \ \mathrm{tr}\left(\frac{\boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g_k}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_3(\boldsymbol{g_k}\cdot\boldsymbol{\tau}) - (\boldsymbol{g_k}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_3(\boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g_k}\cdot\boldsymbol{\tau}))}{g^3}\partial_{k_1} g_3\right) =$$

$$\left\{\begin{array}{c}\text{Here we use the identity} \\ \mathrm{tr}\left(\boldsymbol{\tau}_i\boldsymbol{\tau}_j\boldsymbol{\tau}_k\right) = 2i\epsilon_{ijk}\end{array}\right\} = -\frac{1}{8}\frac{1}{(2\pi)^2}\iint \mathrm{d}^2 k \frac{\epsilon_{ij3}\left[(q_1\partial_{k_1} + q_2\partial_{k_2})g_j\right] g_i \partial_{k_1} g_3}{g^3} =$$

$$-\frac{1}{8}\frac{1}{(2\pi)^2}\iint \mathrm{d}^2 k \frac{q_2(\partial_{k_2} g_2) g_1 \partial_{k_1} g_3 - q_1(\partial_{k_1} g_1) g_2 \partial_{k_1} g_3}{g^3} = \left\{\begin{array}{c}\text{Due to the antisymmetry} \\ \text{with respect to } k_1\end{array}\right\}$$

$$= -\frac{1}{8}\frac{1}{(2\pi)^2}\iint \mathrm{d}^2 k \frac{q_2(\partial_{k_2} g_2) g_1 (\partial_{k_1} g_3)}{g^3} = -\frac{q_2}{32\pi^2}\iint \mathrm{d}^2 k \ \frac{\Delta^2 k_1^2}{m g_{\boldsymbol{k}}^3} = -\frac{q_2}{32\pi}\int_0^\infty dk \ \frac{\Delta^2 k^3}{m g_{\boldsymbol{k}}^3} = -\frac{q_2}{16\pi} \tag{53}$$

$$\Pi^{(1)02}_{\ a}(q) = -\Pi^{(1)20}_{\ a}(q) = \frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\ \mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^0_a \mathcal{G}^{(0)}_k j^2_a + \mathcal{G}^{(0)}_k j^0_a \mathcal{G}^{(1)}_{k,-q} j^2_a\right) =$$

$$-\frac{1}{8}\frac{1}{(2\pi)^3}\iiint \mathrm{d}^3k\ \mathrm{tr}\left(\frac{(q_0\boldsymbol{\tau}_0 - \boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g_k}\cdot\boldsymbol{\tau}))\boldsymbol{\tau}_3(\boldsymbol{\tau}_0 k_0 + \boldsymbol{g_k}\cdot\boldsymbol{\tau})) - (\boldsymbol{\tau}_0 k_0 + \boldsymbol{g_k}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_3(q_0\boldsymbol{\tau}_0 - \boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g_k}\cdot\boldsymbol{\tau}))}{(-k_0^2+g^2)^2}\partial_{k_2}g_3\right) =$$

$$\left\{\begin{array}{c} q_0 \text{ is proportional to} \\ (\boldsymbol{g}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_3 - \boldsymbol{\tau}_3(\boldsymbol{g}\cdot\boldsymbol{\tau}) \\ \text{which has zero trace} \\ \text{and we integrate over } k_0. \end{array}\right\} = \frac{i}{32}\frac{1}{(2\pi)^2}\iint \mathrm{d}^2k\ \mathrm{tr}\left(\frac{\boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g_k}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_3(\boldsymbol{g_k}\cdot\boldsymbol{\tau}) - (\boldsymbol{g_k}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_3(\boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g_k}\cdot\boldsymbol{\tau}))}{g^3}\partial_{k_2}g_3\right) =$$

$$\left\{\begin{array}{c} \text{Here we use the identity} \\ \mathrm{tr}\left(\boldsymbol{\tau}_i\boldsymbol{\tau}_j\boldsymbol{\tau}_k\right) = 2i\epsilon_{ijk} \end{array}\right\} = -\frac{1}{8}\frac{1}{(2\pi)^2}\iint \mathrm{d}^2k\frac{\epsilon_{ij3}\left[(q_1\partial_{k_1} + q_2\partial_{k_2})g_j\right] g_i\partial_{k_2}g_3}{g^3} =$$

$$-\frac{1}{8}\frac{1}{(2\pi)^2}\iint \mathrm{d}^2k\frac{q_2(\partial_{k_2}g_2)g_1\partial_{k_2}g_3 - q_1(\partial_{k_1}g_1)g_2\partial_{k_2}g_3}{g^3} = \left\{\begin{array}{c} \text{Due to the antisymmetry} \\ \text{with respect to } k_2 \end{array}\right\}$$

$$= \frac{1}{8}\frac{1}{(2\pi)^2}\iint \mathrm{d}^2k\frac{q_1(\partial_{k_1}g_1)g_2(\partial_{k_2}g_3)}{g^3} = \frac{q_1}{32\pi^2}\iint \mathrm{d}^2k\ \frac{\Delta^2k_2^2}{mg^3_{\boldsymbol{k}}} = \frac{q_1}{32\pi}\int_0^\infty \mathrm{d}k\ \frac{\Delta^2k^3}{mg^3_{\boldsymbol{k}}} = \frac{q_1}{16\pi} \quad (54)$$

$$\Pi^{(1)00}_{\ a}(q) = \frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\ \mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^0_a \mathcal{G}^{(0)}_k j^0_a + \mathcal{G}^{(0)}_k j^0_a \mathcal{G}^{(1)}_{k,-q} j^0_a\right) = \left\{\begin{array}{c} \text{We use the relation} \\ \mathcal{G}^{(1)}_{k,-q} = -\mathcal{G}^{(1)}_{k,q} \end{array}\right\}$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^0_a \mathcal{G}^{(0)}_k j^0_a - \mathcal{G}^{(0)}_k j^0_a \mathcal{G}^{(1)}_{k,q} j^0_a\right) = \left\{\begin{array}{c} \text{We use the cyclic property of the} \\ \text{trace to make a permutation.} \end{array}\right\} =$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^0_a \mathcal{G}^{(0)}_k j^0_a - \mathcal{G}^{(1)}_{k,q} j^0_a \mathcal{G}^{(0)}_k j^0_a\right) = 0 \quad (55)$$

$$\Pi^{(1)11}_{\ a}(q) = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint \mathrm{d}^3k\ \mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^1_a \mathcal{G}^{(0)}_k j^1_a + \mathcal{G}^{(0)}_k j^1_a \mathcal{G}^{(1)}_{k,-q} j^1_a\right) = \left\{\begin{array}{c} \text{We use the relation} \\ \mathcal{G}^{(1)}_{k,-q} = -\mathcal{G}^{(1)}_{k,q} \end{array}\right\}$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^1_a \mathcal{G}^{(0)}_k j^1_a - \mathcal{G}^{(0)}_k j^1_a \mathcal{G}^{(1)}_{k,q} j^1_a\right) = \left\{\begin{array}{c} \text{We use the cyclic property of the} \\ \text{trace to make a permutation.} \end{array}\right\} =$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^1_a \mathcal{G}^{(0)}_k j^1_a - \mathcal{G}^{(1)}_{k,q} j^1_a \mathcal{G}^{(0)}_k j^1_a\right) = 0 \quad (56)$$

$$\Pi^{(1)22}_{\ a}(q) = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint \mathrm{d}^3k\ \mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^2_a \mathcal{G}^{(0)}_k j^2_a + \mathcal{G}^{(0)}_k j^2_a \mathcal{G}^{(1)}_{k,-q} j^2_a\right) = \left\{\begin{array}{c}\text{We use the relation}\\ \mathcal{G}^{(1)}_{k,-q} = -\mathcal{G}^{(1)}_{k,q}\end{array}\right\}$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\,\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^2_a \mathcal{G}^{(0)}_k j^2_a - \mathcal{G}^{(0)}_k j^2_a \mathcal{G}^{(1)}_{k,q} j^2_a\right) = \left\{\begin{array}{c}\text{We use the cyclic property of the}\\ \text{trace to make a permutation.}\end{array}\right\} =$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\,\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^2_a \mathcal{G}^{(0)}_k j^2_a - \mathcal{G}^{(1)}_{k,q} j^2_a \mathcal{G}^{(0)}_k j^2_a\right) = 0 \quad (57)$$

$$\Pi^{(1)12}_{\ a}(q) = -\Pi^{(1)21}_{\ a} = \frac{1}{4}\frac{1}{(2\pi)^3}\iiint \mathrm{d}^3k\ \mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^1_a \mathcal{G}^{(0)}_k j^2_a + \mathcal{G}^{(0)}_k j^1_a \mathcal{G}^{(1)}_{k,-q} j^2_a\right) = \left\{\begin{array}{c}\text{We use the relation}\\ \mathcal{G}^{(1)}_{k,-q} = -\mathcal{G}^{(1)}_{k,q}\end{array}\right\}$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\,\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^1_a \mathcal{G}^{(0)}_k j^2_a - \mathcal{G}^{(0)}_k j^1_a \mathcal{G}^{(1)}_{k,q} j^2_a\right) = \left\{\begin{array}{c}j^1_a \text{ and } j^2_a \text{ are proportional to the identity matrix}\\ \text{so swapping } G^{(0)} \text{ and } G^{(1)} \text{ wouldn't alter the trace}\end{array}\right\} =$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\,\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^1_a \mathcal{G}^{(0)}_k j^2_a - \mathcal{G}^{(1)}_{k,q} j^1_a \mathcal{G}^{(0)}_k j^2_a\right) = 0 \quad (58)$$

6.2 The correlator Π_b

$$\Pi^{(1)01}_{\;b}(q) = -\Pi^{(1)10}_{\;b}(q) = \frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k \; \mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^0_b \mathcal{G}^{(0)}_k j^1_b + \mathcal{G}^{(0)}_k j^0_b \mathcal{G}^{(1)}_{k,-q} j^1_b\right) =$$

$$-\frac{1}{8}\frac{1}{(2\pi)^3}\iiint \mathrm{d}^3k \; \mathrm{tr}\left(\frac{(q_0\boldsymbol{\tau}_0 - \boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g_k}\cdot\boldsymbol{\tau}))(\boldsymbol{\tau}_0 k_0 + \boldsymbol{g_k}\cdot\boldsymbol{\tau}) - (\boldsymbol{\tau}_0 k_0 + \boldsymbol{g_k}\cdot\boldsymbol{\tau})(q_0\boldsymbol{\tau}_0 - \boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g_k}\cdot\boldsymbol{\tau}))}{(-k_0^2 + g^2)^2}\partial_{k_1}\boldsymbol{g}\cdot\boldsymbol{\tau}\right)$$

$$= \left\{\begin{array}{c}\text{Terms involving } q_0 \text{ cancel each} \\ \text{other,we integrate over } k_0 \text{ and} \\ \text{use the relation } \epsilon_{jik} = -\epsilon_{ijk}\end{array}\right\} = \frac{i}{16}\frac{1}{(2\pi)^2}\iint \mathrm{d}^2k \; \mathrm{tr}\left(\frac{\boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g_k}\cdot\boldsymbol{\tau})\boldsymbol{g_k}\cdot\boldsymbol{\tau}}{g^3}\partial_{k_1}\boldsymbol{g}\cdot\boldsymbol{\tau}\right)$$

$$= \frac{i}{16}\frac{1}{(2\pi)^2}\iint \mathrm{d}^2k \; \mathrm{tr}\left(\frac{\begin{array}{c}\boldsymbol{q}\cdot\boldsymbol{\nabla}(g_3\boldsymbol{\tau}_3)(g_2\boldsymbol{\tau}_2)\partial_{k_1}g_1\boldsymbol{\tau}_1 + \boldsymbol{q}\cdot\boldsymbol{\nabla}(g_2\boldsymbol{\tau}_2)(g_3\boldsymbol{\tau}_3)\partial_{k_1}g_1\boldsymbol{\tau}_1 \\ +\boldsymbol{q}\cdot\boldsymbol{\nabla}(g_1\boldsymbol{\tau}_1)(g_2\boldsymbol{\tau}_2)\partial_{k_1}g_3\boldsymbol{\tau}_3 + \boldsymbol{q}\cdot\boldsymbol{\nabla}(g_2\boldsymbol{\tau}_2)(g_1\boldsymbol{\tau}_1)\partial_{k_1}g_3\boldsymbol{\tau}_3\end{array}}{g^3}\right) =$$

$$-\frac{1}{8}\frac{1}{(2\pi)^2}\iint \mathrm{d}^2k \; \frac{-\boldsymbol{q}\cdot(\boldsymbol{\nabla}g_3)g_2\partial_{k_1}g_1 + \boldsymbol{q}\cdot(\boldsymbol{\nabla}g_2)g_3\partial_{k_1}g_1 + \boldsymbol{q}\cdot(\boldsymbol{\nabla}g_1)g_2\partial_{k_1}g_3 - \boldsymbol{q}\cdot(\boldsymbol{\nabla}g_2)g_1\partial_{k_1}g_3}{g^3} =$$

$$\frac{1}{8}\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} \mathrm{d}^2k \; \frac{q_1(g_2\partial_{k_1}g_3\partial_{k_1}g_2 - g_2\partial_{k_1}g_1\partial_{k_1}g_3) + q_2(g_3\partial_{k_2}g_2\partial_{k_1}g_1 - g_2\partial_{k_2}g_3\partial_{k_1}g_1 - g_1\partial_{k_2}g_2\partial_{k_1}g_3)}{g_{\boldsymbol{k}}^3} =$$

$$-\frac{\Delta^2 q_2}{16\pi}\int_0^\infty |\boldsymbol{k}|\,\mathrm{d}k\frac{\xi_{\boldsymbol{k}} - \frac{\boldsymbol{k}^2}{m}}{g_{\boldsymbol{k}}^3} = \frac{m\Delta^2 q_2}{16\pi}\int_{-\mu}^\infty \mathrm{d}\xi\;\frac{\xi + 2\mu}{(\xi^2 + 2m\Delta^2(\xi+\mu))^{3/2}} = \frac{q_2}{8\pi} \tag{59}$$

$$
\begin{aligned}
&\Pi^{(1)02}_{\;\;b}(q) = -\Pi^{(1)20}_{\;\;b}(q) = \frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\ \mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^0_b \mathcal{G}^{(0)}_k j^2_b + \mathcal{G}^{(0)}_k j^0_b \mathcal{G}^{(1)}_{k,-q} j^2_b\right) = \\
&-\frac{1}{8}\frac{1}{(2\pi)^3}\iiint \mathrm{d}^3k\ \mathrm{tr}\left(\frac{(q_0\boldsymbol{\tau}_0 - \boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau}))(\boldsymbol{\tau}_0 k_0 + \boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau}) - (\boldsymbol{\tau}_0 k_0 + \boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau})(q_0\boldsymbol{\tau}_0 - \boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau})}{(-k_0^2+g^2)^2}\partial_{k_2}\boldsymbol{g}\cdot\boldsymbol{\tau}\right) \\
&= \left\{\begin{array}{c}\text{Terms involving } q_0 \text{ cancel each} \\ \text{other,we integrate over } k_0 \text{ and} \\ \text{use the relation } \epsilon_{jik} = -\epsilon_{ijk}\end{array}\right\} = \frac{i}{16}\frac{1}{(2\pi)^2}\iint \mathrm{d}^2k\ \mathrm{tr}\left(\frac{\boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau})\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau}}{g^3}\partial_{k_2}\boldsymbol{g}\cdot\boldsymbol{\tau}\right) \\
&= \frac{i}{16}\frac{1}{(2\pi)^2}\iint \mathrm{d}^2k\ \mathrm{tr}\left(\frac{\begin{array}{c}\boldsymbol{q}\cdot\boldsymbol{\nabla}(g_3\boldsymbol{\tau}_3)(g_1\boldsymbol{\tau}_1)\partial_{k_2}g_2\boldsymbol{\tau}_2 + \boldsymbol{q}\cdot\boldsymbol{\nabla}(g_1\boldsymbol{\tau}_1)(g_3\boldsymbol{\tau}_3)\partial_{k_2}g_2\boldsymbol{\tau}_2 \\ +\boldsymbol{q}\cdot\boldsymbol{\nabla}(g_1\boldsymbol{\tau}_1)(g_2\boldsymbol{\tau}_2)\partial_{k_2}g_3\boldsymbol{\tau}_3 + \boldsymbol{q}\cdot\boldsymbol{\nabla}(g_2\boldsymbol{\tau}_2)(g_1\boldsymbol{\tau}_1)\partial_{k_2}g_3\boldsymbol{\tau}_3\end{array}}{g^3}\right) = \\
&-\frac{1}{8}\frac{1}{(2\pi)^2}\iint \mathrm{d}^2k\ \frac{+\boldsymbol{q}\cdot(\boldsymbol{\nabla}g_3)g_1\partial_{k_2}g_2 - \boldsymbol{q}\cdot(\boldsymbol{\nabla}g_1)g_3\partial_{k_2}g_2 + \boldsymbol{q}\cdot(\boldsymbol{\nabla}g_1)g_2\partial_{k_2}g_3 - \boldsymbol{q}\cdot(\boldsymbol{\nabla}g_2)g_1\partial_{k_2}g_3}{g^3} = \\
&-\frac{1}{8}\frac{1}{(2\pi)^2}\iint \mathrm{d}^2k\ \frac{q_1(g_1\partial_{k_1}g_3\partial_{k_2}g_2 + g_2\partial_{k_1}g_1\partial_{k_2}g_3 - g_3\partial_{k_1}g_1\partial_{k_2}g_2) + q_2(g_1\partial_{k_2}g_3\partial_{k_2}g_2 - g_1\partial_{k_2}g_2\partial_{k_2}g_3)}{g^3} = \\
&\frac{\Delta^2 q_1}{16\pi}\int_0^\infty |\boldsymbol{k}|dk\,\frac{\xi_{\boldsymbol{k}} - \frac{\boldsymbol{k}^2}{m}}{g^3_{\boldsymbol{k}}} = -\frac{m\Delta^2 q_1}{16\pi}\int_{-\mu}^\infty d\xi\ \frac{\xi + 2\mu}{(\xi^2 + 2m\Delta^2(\xi+\mu))^{3/2}} = -\frac{q_1}{8\pi} \qquad (60)
\end{aligned}
$$

$$\Pi^{(1)12}_{\ \ b}(q) = -\Pi^{(1)21}_{\ \ b}(q) = \frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\ \mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^1_b \mathcal{G}^{(0)}_k j^2_b + \mathcal{G}^{(0)}_k j^1_b \mathcal{G}^{(1)}_{k,-q} j^2_b\right) =$$

$$-\frac{1}{8}\frac{1}{(2\pi)^3}\iiint \mathrm{d}^3k\ \mathrm{tr}\left(\frac{\begin{array}{c}(q_0\boldsymbol{\tau}_0 - \boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g_k}\cdot\boldsymbol{\tau}))\partial_{k_1}(\boldsymbol{g_k}\cdot\boldsymbol{\tau})(\boldsymbol{\tau}_0 k_0 + \boldsymbol{g_k}\cdot\boldsymbol{\tau})\partial_{k_2}(\boldsymbol{g_k}\cdot\boldsymbol{\tau}) \\ -(\boldsymbol{\tau}_0 k_0 + \boldsymbol{g_k}\cdot\boldsymbol{\tau})\partial_{k_1}(\boldsymbol{g_k}\cdot\boldsymbol{\tau})(q_0\boldsymbol{\tau}_0 - \boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g_k}\cdot\boldsymbol{\tau}))\partial_{k_2}(\boldsymbol{g_k}\cdot\boldsymbol{\tau})\end{array}}{(-k_0^2+g^2)^2}\right) =$$

$$\left\{\begin{array}{c}\text{We exploit the cyclic} \\ \text{property of the trace} \\ \text{and integrate over } k_0\end{array}\right\} = -\frac{i}{32}\frac{1}{(2\pi)^2}\iint \mathrm{d}^2k\ \mathrm{tr}\left(\frac{\begin{array}{c}(\boldsymbol{g_k}\cdot\boldsymbol{\tau})\partial_{k_2}(\boldsymbol{g_k}\cdot\boldsymbol{\tau})(q_0\boldsymbol{\tau}_0 - \boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g_k}\cdot\boldsymbol{\tau}))\partial_{k_1}(\boldsymbol{g_k}\cdot\boldsymbol{\tau}) \\ -(\boldsymbol{g_k}\cdot\boldsymbol{\tau})\partial_{k_1}(\boldsymbol{g_k}\cdot\boldsymbol{\tau})(q_0\boldsymbol{\tau}_0 - \boldsymbol{q}\cdot\boldsymbol{\nabla}(\boldsymbol{g_k}\cdot\boldsymbol{\tau}))\partial_{k_2}(\boldsymbol{g_k}\cdot\boldsymbol{\tau})\end{array}}{g^3}\right) =$$

$$\left\{\begin{array}{c}\text{We use the relations } \epsilon_{ijk} = -\epsilon_{ikj} \\ \text{and } \mathrm{tr}\left(\boldsymbol{\tau}_i\boldsymbol{\tau}_j\boldsymbol{\tau}_k\boldsymbol{\tau}_l\right) = \mathrm{tr}\left(\boldsymbol{\tau}_i\boldsymbol{\tau}_l\boldsymbol{\tau}_k\boldsymbol{\tau}_j\right)\end{array}\right\} = -\frac{i}{16}\frac{q_0}{(2\pi)^2}\int_{\mathbb{R}^2}\mathrm{d}^2k\ \mathrm{tr}\left(\frac{(\boldsymbol{g}\cdot\boldsymbol{\tau})\partial_{k_2}(\boldsymbol{g}\cdot\boldsymbol{\tau})\partial_{k_1}(\boldsymbol{g}\cdot\boldsymbol{\tau})}{g^3_{\boldsymbol{k}}}\right) =$$

$$\frac{1}{8}\frac{q_0}{(2\pi)^2}\int_{\mathbb{R}^2}\mathrm{d}^2k\ \frac{g_1(\partial_{k_2}g_2)(\partial_{k_1}g_3) - g_3(\partial_{k_2}g_2)(\partial_{k_1}g_1) + g_2(\partial_{k_2}g_3)(\partial_{k_1}g_1)}{g^3_{\boldsymbol{k}}} =$$

$$\frac{1}{8}\frac{q_0}{(2\pi)^2}\int_{\mathbb{R}^2}\mathrm{d}^2k\ \frac{\Delta^2\frac{k_1^2}{m} - \xi_{\boldsymbol{k}}\Delta^2 + \Delta^2\frac{k_2^2}{m}}{g^3_{\boldsymbol{k}}} = -\frac{1}{8}\frac{q_0\Delta^2}{(2\pi)^2}\int_{\mathbb{R}^2}\mathrm{d}^2k\ \frac{\xi_{\boldsymbol{k}} - \frac{\boldsymbol{k}^2}{m}}{g^3_{\boldsymbol{k}}} = \frac{q_0}{8\pi} \quad (61)$$

$$\Pi^{(1)00}_{\ \ b}(q) = \frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^0_b \mathcal{G}^{(0)}_k j^0_b + \mathcal{G}^{(0)}_k j^0_b \mathcal{G}^{(1)}_{k,-q} j^0_b\right) = \left\{\begin{array}{c}\text{We use the relation} \\ \mathcal{G}^{(1)}_{k,-q} = -\mathcal{G}^{(1)}_q\end{array}\right\}$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^0_b \mathcal{G}^{(0)}_k j^0_b - \mathcal{G}^{(0)}_k j^0_b \mathcal{G}^{(1)}_{k,q} j^0_b\right) = \left\{\begin{array}{c}\text{We use the cyclic property of the} \\ \text{trace to make a permutation.}\end{array}\right\} =$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^0_b \mathcal{G}^{(0)}_k j^0_b - \mathcal{G}^{(1)}_{k,q} j^0_b \mathcal{G}^{(0)}_k j^0_b\right) = 0 \quad (62)$$

$$\Pi^{(1)11}_{\ \ b}(q) = \frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^1_b \mathcal{G}^{(0)}_k j^1_b + \mathcal{G}^{(0)}_k j^1_b \mathcal{G}^{(1)}_{k,-q} j^1_b\right) = \left\{\begin{array}{c}\text{We use the relation} \\ \mathcal{G}^{(1)}_{k,-q} = -\mathcal{G}^{(1)}_q\end{array}\right\}$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^1_b \mathcal{G}^{(0)}_k j^1_b - \mathcal{G}^{(0)}_k j^1_b \mathcal{G}^{(1)}_{k,q} j^1_b\right) = \left\{\begin{array}{c}\text{We use the cyclic property of the} \\ \text{trace to make a permutation.}\end{array}\right\} =$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j^1_b \mathcal{G}^{(0)}_k j^1_b - \mathcal{G}^{(1)}_{k,q} j^1_b \mathcal{G}^{(0)}_k j^1_b\right) = 0 \quad (63)$$

$$\Pi_b^{(1)22}(q) = \frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\,\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j_b^2 \mathcal{G}^{(0)}_k j_b^2 + \mathcal{G}^{(0)}_k j_b^2 \mathcal{G}^{(1)}_{k,-q} j_b^2\right) = \left\{\begin{array}{c}\text{We use the relation}\\ \mathcal{G}^{(1)}_{k,-q} = -\mathcal{G}^{(1)}_q\end{array}\right\}$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\,\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j_b^2 \mathcal{G}^{(0)}_k j_b^2 - \mathcal{G}^{(0)}_k j_b^2 \mathcal{G}^{(1)}_{k,q} j_b^2\right) = \left\{\begin{array}{c}\text{We use the cyclic property of the}\\ \text{trace to make a permutation.}\end{array}\right\} =$$

$$\frac{1}{4}\frac{1}{(2\pi)^3}\int \mathrm{d}^3k\,\mathrm{tr}\left(\mathcal{G}^{(1)}_{k,q} j_b^2 \mathcal{G}^{(0)}_k j_b^2 - \mathcal{G}^{(1)}_{k,q} j_b^2 \mathcal{G}^{(0)}_k j_b^2\right) = 0 \quad (64)$$

7 Summary

At last, the complete effective action[3] in terms of (2+1)-dimensional frequency-momentum[4], $q = (\boldsymbol{q}, q_0)$ and $k = (\boldsymbol{k}, k_0)$ is:

$$\mathcal{S}(\boldsymbol{q}, q_0) = \int \tfrac{d\boldsymbol{q}}{df}\, n a_0(q)\delta(q) - \frac{n}{2m} a_\mu(q) a_\mu(-q) + a_\mu(q)\pi_a^{\mu\nu} a_\nu(-q) + b_\mu(q)\pi_b^{\mu\nu} b_\nu(-q) \quad (65)$$

where the correlation matrices in the lowest order of the perturbative expansion are written as

$$\Pi_a^{\mu\nu}(q) = \frac{1}{4}\frac{1}{(2\pi)^3}(1 + q\cdot\boldsymbol{\nabla}_{q'})\left[\int d^3k\ \mathrm{tr}\left(\mathcal{G}(k+\frac{q'}{2}) j_a^\mu(\boldsymbol{k})\mathcal{G}(k-\frac{q'}{2}) j_a^\nu(\boldsymbol{k})\right)\right]\Bigg|_{q'=0} = \quad (66)$$

$$= \begin{pmatrix} \frac{m}{4\pi} & -\frac{i\,q_y}{16\pi} & \frac{i\,q_x}{16\pi} \\ \frac{i\,q_y}{16\pi} & 0 & 0 \\ -\frac{i\,q_x}{16\pi} & 0 & 0 \end{pmatrix}\left(1 + \frac{2\mu}{m\Delta^2}H(-\mu)\right)^{-1} + O(\boldsymbol{q}^2)$$

$$\Pi_b^{\mu\nu}(q) = \frac{1}{4}\frac{1}{(2\pi)^3} q\cdot\boldsymbol{\nabla}_{q'}\left[\int d^3k\ \mathrm{tr}\left(\mathcal{G}(k+\frac{q'}{2}) j_b^\mu(\boldsymbol{k})\mathcal{G}(k-\frac{q'}{2}) j_b^\nu(\boldsymbol{k})\right)\right]\Bigg|_{q'=0} = \quad (67)$$

$$= \begin{pmatrix} 0 & \frac{i\,q_y}{8\pi} & -\frac{i\,q_x}{8\pi} \\ -\frac{i\,q_y}{8\pi} & 0 & \frac{i\,f}{8\pi} \\ \frac{i\,q_x}{8\pi} & -\frac{i\,f}{8\pi} & 0 \end{pmatrix} H(\mu) + O(\boldsymbol{q}^2)$$

and currents for the fields $\boldsymbol{a}$ and $\boldsymbol{b}$ are

$$\mathbf{j}_a = (-\boldsymbol{\tau}_3, -\tfrac{k_x}{m}\boldsymbol{\tau}_0, -\tfrac{k_y}{m}\boldsymbol{\tau}_0), \quad \mathbf{j}_b = -\boldsymbol{\tau}_3\mathbf{j}^a + (0, \Delta\boldsymbol{\tau}_1, \Delta\boldsymbol{\tau}_2) \quad (68)$$

transforming back into space-time coordinates yields

$$\mathcal{S}(\mathbf{x}, t) = \int \tfrac{d\boldsymbol{x}}{dt}\left(n\left(a_0 - \frac{1}{2m}\boldsymbol{a}^2\right) + \frac{m}{4\pi}a_0^2 - \frac{\kappa_a}{8\pi}\epsilon_{0jk} a_0 \partial_j a_k + \frac{\kappa_b}{8\pi}\epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda\right) \quad (69)$$

[3] We extended the result for the case of negative chemical potential.

Where $\boldsymbol{a} = \mathbf{A} - \partial_{\boldsymbol{x}}\theta/2$, $a_0 = A_0 - \partial_t\theta/2$, $\boldsymbol{b} = \partial_{\boldsymbol{x}}\gamma$, $b_t = \partial_t\gamma$, $\kappa_a = \left(1 + \frac{2\mu}{m\Delta^2}H(-\mu)\right)^{-1}$, $\kappa_b = H(\mu)$ and $\Delta_0 = 2|\Delta|$. The first term in the r.h.s contributes the Magnus force, the second gives rise to the Meissner effect, and the third is responsible to the Thomas-Fermi screening. The forth term is an incomplete Chern-Simons term that contributes a Hall-like response to the external field.

[4]Here we define $q = (\boldsymbol{q}, q_0)$ and not as was used in the derivation of the correlators,$q = (\boldsymbol{q}, -q_0)$.

Appendices

A Multidimensional Gaussian integration for real Grassmann fields

In the appendix we show how to calculate a multidimensional Gaussian integrals over Grassmann variables, η_k,

$$\int \left(\prod_k d\eta_k^* d\eta_k\right) \exp\left(-\frac{1}{2}\eta_k^* A_{kl}\eta_l\right), \tag{70}$$

where A is a Hermitian matrix and there is summation over repeated Latin indices[6, 14]. We start with the observation that A may always be assumed to be skew-symmetric. For if there were any symmetric part, it would cancel due to the anticommutativity of the Grassmann variables,

$$\frac{1}{2}\sum_{kl}\eta_k^*[\mathrm{Sym(A)}]_{kl}\eta_l = \frac{1}{2}\sum_{kl}\eta_k^*(A_{kl}+A_{lk})\eta_l = \frac{1}{2}\sum_{k>l}(A_{kl}+A_{lk})(\eta_k^*\eta_l+\eta_l\eta_k^*) = 0. \tag{71}$$

Now, an skew-symmetric Hermitian matrix can always be written as $A = iA'$ where A' is real skew-symmetric. Since the eigenvalues of a real skew-symmetric matrix are imaginary, the diagonalization can only be carried by a complex unitary matrix. In general, similarity transformations by unitary matrices should be avoided, since it would alter the measure of integration $\prod_k d\eta_k^* d\eta_k$. However, it is possible to bring every real skew-symmetric matrix into a canonical form, $\tilde{A} = U^T A' U$, where U is a special orthogonal matrix,

$$\tilde{A} = \bigoplus_{j=1}^{n}\begin{pmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{pmatrix},$$

$2n$ is the dimension of A and $\pm i\lambda_j$ are the eigenvalues of A.

The Jacobian determinant, due to the linear transformation $\xi_k = (U^T)_{kl}\eta_l$ of the integration variables, is always unity ($\det U = 1$), so the integration measure is invariant, $\prod_k d\eta_k^* d\eta_k = \prod_k d\xi_k^* d\xi_k$. Thus, applying a block diagonalization transformation yields and evaluating the Gaussian integral,

$$\int \left(\prod_k d\eta_k^* d\eta_k\right) e^{-\frac{1}{2}\eta_m^* A_{mn}\eta_n} = \int \left(\prod_k d\xi_k^* d\xi_k\right) e^{-\frac{i}{2}\xi_k^* \tilde{A}_{kl}\xi_l} =$$
$$\int \left(\prod_k d\xi_k^* d\xi_k\right)\left(1 - \frac{i}{2}\xi_k^*\tilde{A}_{kl}\xi_l\right) = \prod_{k=1}^{n} i\lambda_k = \mathrm{pf}(i\tilde{A}) = \mathrm{pf}(U^T A U) = \mathrm{pf}(A), \tag{72}$$

where we used the relations $\mathrm{pf}(U^T A U) = \det(U)\mathrm{pf}(A)$ and $\mathrm{pf}(\oplus_n A_n) = \prod_i \mathrm{pf}(A_n)$.

Since $\det A = (\mathrm{pf} A)^2$, in the special case that A itself is a block-diagonal matrix, we can write the relation

$$\mathrm{pf}(A) = \pm\prod_n \det(A_n)^{1/2} = \pm\exp\left[\sum_n \frac{1}{2}\mathrm{Tr}\log A_n\right], \tag{73}$$

where A_n are the block matrices on the diagonal of A. Finally, we can write (up to a sign) the result of integration over the real Grassmann fields as:

$$\int \left(\prod_k d\eta_k^* d\eta_k \right) \exp\left(-\frac{1}{2} \sum_{kl} \eta_k^* A_{kl} \eta_l \right) = \prod_n (\det A_n)^{1/2} = \exp\left(\sum_n \frac{1}{2} \mathrm{Tr} \log A_n \right). \tag{74}$$

B Fourier Transform identities based on Parseval's theorem

1. Let $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ be integrable and let $f(\boldsymbol{k})$ and $g(\boldsymbol{k})$ be their Fourier Transform. If $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ are also square-integrable, then we have Parseval's theorem (Rudin 1987,p.187):

$$\int_{-\infty}^{+\infty} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\mathrm{d}\boldsymbol{x} = \int_{-\infty}^{+\infty} f(\boldsymbol{k})\overline{g(\boldsymbol{k})}\mathrm{d}\boldsymbol{k} \tag{75}$$

   In particular, if $g(\boldsymbol{x})$ is real, using the reality condition $g(-\boldsymbol{k}) = \overline{g(\boldsymbol{k})}$, Parseval's theorem takes the form of:

$$\int_{-\infty}^{+\infty} f(\boldsymbol{x})g(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{-\infty}^{+\infty} f(\boldsymbol{k})g(-\boldsymbol{k})\mathrm{d}\boldsymbol{k} \tag{76}$$

   Also, if $f(\boldsymbol{x}) = g(\boldsymbol{x})$ and real we get:

$$\int_{-\infty}^{+\infty} f^2(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{-\infty}^{+\infty} f(\boldsymbol{k})f(-\boldsymbol{k})\mathrm{d}\boldsymbol{k} \tag{77}$$

2. Substituting $\boldsymbol{k} = 0$ in the defenition of the Fourier transform gives

$$f(\boldsymbol{0}) = \int_{-\infty}^{+\infty} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{78}$$

C Identities of Pauli matrices products

We would like to evaluate the trace of four Pauli matrices product. In order to do so, we distinguish between six possible conditions for the matrices combination:

Condition	Implication
$i=j$	$\mathrm{tr}(\boldsymbol{\tau}_i\boldsymbol{\tau}_j\boldsymbol{\tau}_k\boldsymbol{\tau}_l) = \delta_{ij}\mathrm{tr}(\boldsymbol{\tau}_k\boldsymbol{\tau}_l) = 2\delta_{ij}\delta_{kl}$
$i=l$	$\mathrm{tr}(\boldsymbol{\tau}_i\boldsymbol{\tau}_j\boldsymbol{\tau}_k\boldsymbol{\tau}_l) = \delta_{il}\mathrm{tr}(\boldsymbol{\tau}_j\boldsymbol{\tau}_k) = 2\delta_{il}\delta_{jk}$
$i=k$	$\mathrm{tr}(\boldsymbol{\tau}_i\boldsymbol{\tau}_j\boldsymbol{\tau}_k\boldsymbol{\tau}_l) = \delta_{ik}\left(2\delta_{ij}\mathrm{tr}(\boldsymbol{\tau}_k\boldsymbol{\tau}_l) - \mathrm{tr}(\boldsymbol{\tau}_j\boldsymbol{\tau}_i\boldsymbol{\tau}_k\boldsymbol{\tau}_l)\right) = 4\delta_{ij}\delta_{kl}\delta_{ik} - 2\delta_{ik}\delta_{jl}$
$j=k$	$\mathrm{tr}(\boldsymbol{\tau}_i\boldsymbol{\tau}_j\boldsymbol{\tau}_k\boldsymbol{\tau}_l) = \delta_{jk}\mathrm{tr}(\boldsymbol{\tau}_l\boldsymbol{\tau}_i) = 2\delta_{jk}\delta_{il}$
$j=l$	$\mathrm{tr}(\boldsymbol{\tau}_i\boldsymbol{\tau}_j\boldsymbol{\tau}_k\boldsymbol{\tau}_l) = \delta_{jl}\left(2\delta_{ij}\mathrm{tr}(\boldsymbol{\tau}_k\boldsymbol{\tau}_l) - \mathrm{tr}(\boldsymbol{\tau}_j\boldsymbol{\tau}_i\boldsymbol{\tau}_k\boldsymbol{\tau}_l)\right) = 4\delta_{jl}\delta_{ij}\delta_{kl} - 2\delta_{jl}\delta_{ik}$
$k=l$	$\mathrm{tr}(\boldsymbol{\tau}_i\boldsymbol{\tau}_j\boldsymbol{\tau}_k\boldsymbol{\tau}_l) = \delta_{kl}\mathrm{tr}(\boldsymbol{\tau}_i\boldsymbol{\tau}_j) = 2\delta_{kl}\delta_{ij}$

Where δ_{ab} is Kronecker's delta. In order to evaluate the traces in the table we used the identities $\{\boldsymbol{\tau}_a, \boldsymbol{\tau}_b\} = 2\delta_{ab}\mathbf{I}$ and $\mathrm{tr}(\boldsymbol{\tau}_a\boldsymbol{\tau}_b) = 2\delta_{ab}$.

Summing up all the different possible implication yields:

$$\mathrm{tr}\left(\boldsymbol{\tau}_\mathbf{i}\boldsymbol{\tau}_\mathbf{j}\boldsymbol{\tau}_\mathbf{k}\boldsymbol{\tau}_\mathbf{l}\right) = 2\delta_{ij}\delta_{kl} + 2\delta_{il}\delta_{jk} - 2\delta_{ik}\delta_{jl} \tag{79}$$

If one matrices in the combination is the identity matrix then we have a sum of three Pauli matrices. In this case it is easy to show that

$$\mathrm{tr}\left(\boldsymbol{\tau}_\mathbf{i}\boldsymbol{\tau}_\mathbf{j}\boldsymbol{\tau}_\mathbf{k}\right) = \epsilon_{ijk}2i \tag{80}$$

where ϵ_{ijk} is the Levi-Civita symbol.

D Fundamental quantities in Natural units

In the notes we used Natural units where $c = \hbar = e = 1$ are unitless. We summarize the fundamental quantities units under this unit system:

Quantity	Symbol	SI units	$\rightarrow$	Natural Units	Relation
Length	x	(m)	$\rightarrow$	(s)	$x = ct$
Mass	m	(kg)	$\rightarrow$	(s^{-1})	$mc^2 = \hbar\omega$
Density	n	(m^{-2})	$\rightarrow$	(s^{-2})	$n = \#\boldsymbol{k}^2$
Momentum	p	$(kg\ m/s)$	$\rightarrow$	(s^{-1})	$p = mv$
Energy	E	(J)	$\rightarrow$	(s^{-1})	$E = \hbar\omega$
Action	S	$(J \cdot\ s)$	$\rightarrow$	(1)	$Z = e^{-S/\hbar}$
Scalar Potential	a_0	(J/C)	$\rightarrow$	(s^{-1})	$E = e\ a_0$
Vector Potential	$\mathbf{a}$	$(\frac{kg\ m/s}{C})$	$\rightarrow$	(s^{-1})	$E = \frac{(\mathbf{p}-e\mathbf{a})^2}{2m}$
Gap	Δ	(m/s)	$\rightarrow$	(1)	$E = \Delta p$
Gauge fields	b,c,d	(kg m/s)	$\rightarrow$	(s^{-1})	$E = \frac{\mathbf{b}^2+\mathbf{c}^2+\mathbf{d}^2}{2m}$
Flux Quanta	Φ_0	(J s/C)	$\rightarrow$	(1)	$\Phi_0 = \frac{hc}{e}$
Current density	j	(C m^{-2} m/s)	$\rightarrow$	(s^{-2})	$j = env$

E Derivation of the gauge transformation

In this section we derive the most general gauge transformation that maintain the requirements as stated in section 2.

The first rule over the transformation **U**, as demanded in section 2, is:

$$\mathbf{U} \cdot \eta_{\mathbf{x},t} = \left[\bar{\eta}_{\mathbf{x},t} \cdot \mathbf{U}^{-1}\right]^\dagger = (\mathbf{U}^{-1})^\dagger \cdot \eta_{\mathbf{x},t} \tag{81}$$

which gives the first demand over the transformation:

$$\mathbf{U}^{-1} = \mathbf{U}^\dagger \tag{82}$$

The Nambu spinors that correspond to BdG Hamiltonian,as derived in Eq.(4), fulfill the particle-hole symmetry so they hold the form:

$$\eta_{\boldsymbol{x},t} = \begin{pmatrix} \phi_{\boldsymbol{x},t} \\ \bar{\phi}_{\boldsymbol{x},t} \end{pmatrix} \text{ and } \bar{\eta}_{\boldsymbol{x},t} = \begin{pmatrix} \bar{\phi}_{\boldsymbol{x},t}, \phi_{\boldsymbol{x},t} \end{pmatrix} \tag{83}$$

where we define $\bar{\eta} \equiv \eta^\dagger$.
Under the the particle-hole symmetry it can be also written as

$$\bar{\eta} = \eta^T \boldsymbol{\tau}_1 \tag{84}$$

where $\boldsymbol{\tau}_1$ is the first Pauli matrix.

Using the transformation, **U**, we obtain a new set of 2D spinors which are denoted by

$$\Psi = \mathbf{U}\eta, \qquad \bar{\Psi} = (\mathbf{U}\eta)^\dagger = \eta^\dagger \mathbf{U}^\dagger, \tag{85}$$

where $\bar{\Psi}$ is obtained directly from the definition.

According to the second requirement, the transformation **U** must sustain the form of the spinors so

$$\bar{\Psi} = (\mathbf{U}\eta)^T \boldsymbol{\tau}_1 = \eta^T \mathbf{U}^T \boldsymbol{\tau}_1 = \eta^\dagger \boldsymbol{\tau}_1 \mathbf{U}^T \boldsymbol{\tau}_1 \tag{86}$$

Equating equations (85) and (86) yields the second condition over **U**:

$$\mathbf{U}^\dagger = \boldsymbol{\tau}_1 \mathbf{U}^T \boldsymbol{\tau}_1 \tag{87}$$

The first requirement implies that the transformation should be represented by unitary matrix. Any unitary matrix can be represented by a matrix exponential of the form -

$$\mathbf{U} = e^{i(\alpha\boldsymbol{\tau}_0 + \beta\boldsymbol{\tau}_1 + \gamma\boldsymbol{\tau}_2 + \delta\boldsymbol{\tau}_3)} \tag{88}$$

since it satisfies $\mathbf{U} \cdot \mathbf{U}^\dagger = \boldsymbol{\tau}_0$ and has four degrees of freedom (it can produce four linear independent matrices).

Before continuing we recall that matrix exponential fulfill the following properties:

$$e^{\mathbf{0}} = \boldsymbol{\tau}_0 \tag{89}$$

$$e^{a\mathbf{X}} e^{b\mathbf{X}} = e^{(a+b)\mathbf{X}} \tag{90}$$

$$e^{\mathbf{X}} e^{-\mathbf{X}} = \boldsymbol{\tau}_0 \tag{91}$$

$$e^{\mathbf{Y}\mathbf{X}\mathbf{Y}^{-1}} = \mathbf{Y} e^{\mathbf{X}} \mathbf{Y}^{-1} \tag{92}$$

$$e^{(\mathbf{X}^T)} = (e^{\mathbf{X}})^T \tag{93}$$

$$e^{(\mathbf{X}^\dagger)} = (e^{\mathbf{X}})^\dagger \tag{94}$$

Where **X** and **Y** are n×n complex matrices, a and b are arbitrary complex numbers. We denote the n×n identity matrix by $\boldsymbol{\tau}_0$ and the zero matrix by $\mathbf{0}$. Another handy identity is the anticommutation property of the Pauli matrices

$$\{\boldsymbol{\tau}_i, \boldsymbol{\tau}_j\} = 2\delta_{ij} \tag{95}$$

Now, armed with this knowledge we continue by finding the constrains on the general unitary matrix express as matrix exponential with four parameters. Substituting Eq.(88) in the right side of the equality in Eq.(87) yields

$$\boldsymbol{\tau}_1 \mathbf{U}^T \boldsymbol{\tau}_1 = e^{i\boldsymbol{\tau}_1(\alpha\boldsymbol{\tau}_0^T + \beta\boldsymbol{\tau}_1^T + \gamma\boldsymbol{\tau}_2^T + \delta\boldsymbol{\tau}_3^T)\boldsymbol{\tau}_1} = e^{i\boldsymbol{\tau}_1(\alpha\boldsymbol{\tau}_0 + \beta\boldsymbol{\tau}_1 - \gamma\boldsymbol{\tau}_2 + \delta\boldsymbol{\tau}_3)\boldsymbol{\tau}_1} = e^{i(\alpha\boldsymbol{\tau}_0 + \beta\boldsymbol{\tau}_1 + \gamma\boldsymbol{\tau}_2 - \delta\boldsymbol{\tau}_3)} \tag{96}$$

Making the substitution on the other side gives

$$\mathbf{U}^\dagger = e^{-i(\alpha\boldsymbol{\tau}_0^\dagger + \beta\boldsymbol{\tau}_1^\dagger + \gamma\boldsymbol{\tau}_2^\dagger + \delta\boldsymbol{\tau}_3^\dagger)} = e^{-i(\alpha\boldsymbol{\tau}_0 + \beta\boldsymbol{\tau}_1 + \gamma\boldsymbol{\tau}_2 + \delta\boldsymbol{\tau}_3)} \tag{97}$$

So the constrain over the parameters is

$$\begin{aligned}
\boldsymbol{\tau}_0 \quad = \quad & (\mathbf{U}^\dagger)^{-1} \cdot (\boldsymbol{\tau}_1 \mathbf{U}^T \boldsymbol{\tau}_1) = e^{i(\alpha\boldsymbol{\tau}_0 + \beta\boldsymbol{\tau}_1 + \gamma\boldsymbol{\tau}_2 + \delta\boldsymbol{\tau}_3)} \cdot e^{i(\alpha\boldsymbol{\tau}_0 + \beta\boldsymbol{\tau}_1 + \gamma\boldsymbol{\tau}_2 - \delta\boldsymbol{\tau}_3)} = \qquad (98) \\
& e^{i\alpha}(\boldsymbol{\tau}_0 \cos r + i\hat{\mathbf{r}} \cdot \boldsymbol{\tau} \sin r) \cdot e^{i\alpha}(\boldsymbol{\tau}_0 \cos r + i(\hat{\mathbf{r}} - \frac{2\delta}{r}\hat{\boldsymbol{k}}) \cdot \boldsymbol{\tau} \sin r) = \\
& e^{i2\alpha}\left((\boldsymbol{\tau}_0 \cos r + i\hat{\mathbf{r}} \cdot \boldsymbol{\tau} \sin r)^2 - 2i(\boldsymbol{\tau}_0 \cos r + i\hat{\mathbf{r}} \cdot \boldsymbol{\tau} \sin r)\frac{\delta}{r} \sin r \; \boldsymbol{\tau}_3\right) = \\
& e^{i2\alpha}\left(\cos^2 r - \sin^2 r + i\frac{\beta\boldsymbol{\tau}_1 + \gamma\boldsymbol{\tau}_2 + \delta\boldsymbol{\tau}_3}{r} \sin 2r - i\frac{\delta\boldsymbol{\tau}_3}{r} \sin 2r + 2\frac{\delta}{r}\left(i\frac{-\beta\boldsymbol{\tau}_2 + \gamma\boldsymbol{\tau}_1}{r} + \frac{\delta\boldsymbol{\tau}_0}{r}\right) \sin^2 r\right) \\
& e^{i2\alpha}\left(\left(\frac{\beta^2 + \gamma^2}{r^2} \cos 2r + \frac{\delta^2}{r^2}\right)\boldsymbol{\tau}_0 + i\frac{\beta\boldsymbol{\tau}_1 + \gamma\boldsymbol{\tau}_2}{r} \sin 2r - 2\delta i\frac{\beta\boldsymbol{\tau}_2 - \gamma\boldsymbol{\tau}_1}{r^2} \sin^2 r\right) \qquad (99)
\end{aligned}$$

In this this derivation the identities below were used

$$e^{i\alpha + ir(\hat{\mathbf{r}} \cdot \boldsymbol{\tau})} = e^{i\alpha}(\boldsymbol{\tau}_0 \cos r + i(\hat{\mathbf{r}} \cdot \boldsymbol{\tau}) \sin r), \qquad \boldsymbol{\tau}_i \boldsymbol{\tau}_j = i\epsilon_{ijk}\boldsymbol{\tau}_k + \delta_{ij}\boldsymbol{\tau}_0 \tag{100}$$

where $\boldsymbol{\tau} = (\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, , \boldsymbol{\tau}_3)$ and $\mathbf{r} = (\beta, \gamma, \delta)$.

We start by examine the diagonal elements, which are required to be equal to 1, so:

$$e^{i2\alpha}\left((\beta^2 + \gamma^2) \cos 2r + \delta^2\right) = \beta^2 + \gamma^2 + \delta^2 \tag{101}$$

Since the diagonal elements of $\mathbf{U}$ must be real, one must require that

$$e^{i2\alpha} = \pm 1 \tag{102}$$

Thus, this implies two possible solutions:

1. For $\alpha = \pi m$ and $m \in \mathbb{Z}$, we get that

$$(\beta^2 + \gamma^2) \cos 2r = \beta^2 + \gamma^2 \tag{103}$$

so one option is to require that $\beta^2 + \gamma^2 = 0$. Also, in order to keep zero off-diagonal elements $\gamma = 0$. In this case the transformation is given by

$$\mathbf{U} = e^{i(\pi m + \boldsymbol{\tau}_3 \delta)} = e^{i\pi m}(\boldsymbol{\tau}_0 \cos \delta + i\boldsymbol{\tau}_3 \sin \delta) \tag{104}$$

2. For $\alpha = \pi m$ and $m \in \mathbb{Z}$, we find that

$$(\beta^2 + \gamma^2)\cos 2r = \beta^2 + \gamma^2 \tag{105}$$

the second option is to demand that $r = \pi n$ and $n \in \mathbb{Z}$. These demands keep the off-diagonal elements zero. In this case the transformation is given by

$$\mathbf{U}_a = e^{i\pi m}(\boldsymbol{\tau}_0 \cos \pi n + i\hat{\mathbf{r}} \cdot \boldsymbol{\tau} \sin \pi n) \tag{106}$$

but the second term, which proportional to sinus, is always zero so we end up with

$$\mathbf{U}_a = e^{i\pi m}\boldsymbol{\tau}_0 \cos \pi n \tag{107}$$

For $\alpha = \frac{\pi}{2}(2m - 1)$ and $m \in \mathbb{Z}$, we find that

$$(\beta^2 + \gamma^2)(1 + \cos 2r) = -2\delta^2 \tag{108}$$

and since the left side of equality is non-negative, one must choose $\delta = 0$, $r = \frac{\pi}{2}(2n + 1)$ and $n \in \mathbb{Z}$. These demands also fulfill the requirements of zero off-diagonal elements. In this case the transformation is given by

$$\mathbf{U}_b = e^{i\frac{\pi}{2}(2m-1)}\left(\boldsymbol{\tau}_0 \cos\left(\frac{\pi}{2}(2n+1)\right) + i\hat{\mathbf{r}} \cdot \boldsymbol{\tau} \sin\left(\frac{\pi}{2}(2n+1)\right)\right) \tag{109}$$

but the first term, which proportional to cosines, is always zero so we end up with

$$\mathbf{U}_b = e^{i\pi m}\hat{\mathbf{r}} \cdot \boldsymbol{\tau} \sin\left(\frac{\pi}{2}(2n+1)\right) \tag{110}$$

The two different options can be combined into one transformation

$$\mathbf{U} = e^{i\pi m}\left(\boldsymbol{\tau}_0 \cos\frac{\pi}{2}n + \hat{\mathbf{r}} \cdot \boldsymbol{\tau} \sin\frac{\pi}{2}n\right) \tag{111}$$

where $\{n, m\} \in \mathbb{Z}$, $\boldsymbol{\tau} = (\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ and $\hat{\mathbf{r}} = (\cos\zeta, \sin\zeta)$.

One can readily show that any $\mathbf{U}$ is composed of a finite product of the following matrices: $\tau_x, \tau_y, e^{i\mu\tau_z}$ and $e^{i\pi m}\boldsymbol{\tau}_0$ where $\mu \in \mathbb{R}$ and $m \in \mathbb{Z}$. The actual number of distinct sequences can be reduced through use of the commutations relations between the generators and is ultimately finite.

F Applying the single-valued transformation to the inverse Green's matrix

In this section we apply transformation

$$\mathbf{U} = \lambda e^{-i\tau_z \theta(\boldsymbol{x},t)}, \quad \lambda = e^{-i\gamma(\boldsymbol{x},t)} \tag{112}$$

over the 2D p-wave Green's matrix

$$\mathcal{G}^{-1} = \begin{pmatrix} i\partial_t + A_0 - \frac{1}{2m}(-\boldsymbol{p} + \mathbf{A})^2 + \mu & -\left\{\Delta, \boldsymbol{p}_x - i\boldsymbol{p}_y\right\} \\ -\left\{\overline{\Delta}, \boldsymbol{p}_x + i\boldsymbol{p}_y\right\} & i\partial_t - A_0 + \frac{1}{2m}(\boldsymbol{p} + \mathbf{A})^2 - \mu \end{pmatrix}, \Delta = \frac{\Delta_0}{2} e^{i2\theta(\boldsymbol{x},t)} \tag{113}$$

Thus, we need to evaluate the expression $\mathbf{U}\mathcal{G}^{-1}\mathbf{U}^{\dagger}$,

$$\begin{pmatrix} e^{-i(\theta+\gamma)} & 0 \\ 0 & e^{i(\theta-\gamma)} \end{pmatrix} \begin{pmatrix} i\partial_t + A_0 - \frac{1}{2m}(-\boldsymbol{p} + \mathbf{A})^2 + \mu & -\left\{\Delta, \boldsymbol{p}_x - i\boldsymbol{p}_y\right\} \\ -\left\{\overline{\Delta}, \boldsymbol{p}_x + i\boldsymbol{p}_y\right\} & i\partial_t - A_0 + \frac{1}{2m}(\boldsymbol{p} + \mathbf{A})^2 - \mu \end{pmatrix} \begin{pmatrix} e^{i(\theta+\gamma)} & 0 \\ 0 & e^{-i(\theta-\gamma)} \end{pmatrix}$$

We split the task into a few parts which are calculated individually using the identities

$$-i\partial_{\boldsymbol{x}} g = [\boldsymbol{p}, g], \quad \{\boldsymbol{p}, g\} \equiv \boldsymbol{p}g + g\boldsymbol{p} = 2g\boldsymbol{p} - i\partial_{\boldsymbol{x}} g$$

as follows -

$$\lambda e^{\mp i\theta}(\mp\boldsymbol{p} + \mathbf{A})\bar{\lambda}e^{\pm i\theta} = \mp\boldsymbol{p} \pm i(i\nabla\gamma \pm i\nabla\theta) + \mathbf{A} = \mp(\boldsymbol{p} + \nabla\gamma) + \boldsymbol{a}$$

$$\begin{aligned} \lambda e^{\mp i\theta}(\mp\boldsymbol{p} + \mathbf{A})^2\bar{\lambda}e^{\pm i\theta} &= (\mp\boldsymbol{p} + \boldsymbol{a} \mp \nabla\gamma)^2 = (\mp\boldsymbol{p} + \boldsymbol{a})^2 \mp 2(\nabla\gamma)(\mp\boldsymbol{p} + \boldsymbol{a}) + (\nabla\gamma)^2 - i\nabla^2\gamma \\ &= (\mp\boldsymbol{p} + \boldsymbol{a})^2 + \{\boldsymbol{p}, \nabla\gamma\} \mp 2(\nabla\gamma)\boldsymbol{a} + (\nabla\gamma)^2 \end{aligned} \tag{114}$$

$$\lambda e^{\mp i\theta}\{\frac{\Delta_0}{2}e^{\pm 2i\theta}, \boldsymbol{p}_x\}\bar{\lambda}e^{\mp i\theta} = \lambda e^{\mp i\theta}\frac{\Delta_0}{2}\left(2e^{\pm i\theta}\boldsymbol{p}_x e^{\pm i\theta}\right)\bar{\lambda}e^{\mp i\theta} = \Delta_0\left(\boldsymbol{p}_x + \partial_x\gamma\right) = \Delta_0\boldsymbol{p}_x + \Delta_0\partial_x\gamma$$

where $\{e^{i2\theta}, \boldsymbol{p}_x\} = \boldsymbol{p}_x e^{i\theta}e^{i\theta} + e^{i\theta}e^{i\theta}\boldsymbol{p}_x = e^{i\theta}\boldsymbol{p}_x e^{i\theta} + e^{i2\theta}\partial_x\theta + e^{i\theta}\boldsymbol{p}_x e^{i\theta} - e^{i2\theta}\partial_x\theta = e^{i\theta}2\boldsymbol{p}_x e^{i\theta}$.

So after the transformation the Green's matrix is

$$\mathcal{G}^{-1} = \begin{pmatrix} i\partial_t + a_0 - b_t - \frac{1}{2m}\left[(-\boldsymbol{p} + \boldsymbol{a})^2 + \{\boldsymbol{p}, \boldsymbol{b}\} - 2\boldsymbol{b}\cdot\boldsymbol{a} + \boldsymbol{b}^2\right] + \mu & -\Delta_0(\boldsymbol{p}_x - i\boldsymbol{p}_y) - \Delta_0(b_x - ib_y) \\ -\Delta_0(\boldsymbol{p}_x + i\boldsymbol{p}_y) - \Delta_0(b_x + ib_y) & i\partial_t - a_0 - b_t + \frac{1}{2m}\left[(\boldsymbol{p} + \boldsymbol{a})^2 + \{\boldsymbol{p}, \boldsymbol{b}\} + 2\boldsymbol{b}\cdot\boldsymbol{a} + \boldsymbol{b}^2\right] - \mu \end{pmatrix}$$

In terms of Pauli's matrices it can write the Green's matrix as

$$\begin{aligned} \mathcal{G}^{-1} \;=\; & i\boldsymbol{\tau}_0\partial_t - \boldsymbol{\tau}_3\left(\frac{{\boldsymbol{p}_x}^2 + {\boldsymbol{p}_y}^2}{2m} - \mu\right) - \Delta_0\left(\boldsymbol{p}_x\,\boldsymbol{\tau}_1 + \boldsymbol{p}_y\,\boldsymbol{\tau}_2\right) - \left(-a_0\,\boldsymbol{\tau}_3 - \frac{\{\boldsymbol{p}_x, a_x\} + \{\boldsymbol{p}_y, a_y\}}{2m}\boldsymbol{\tau}_0\right) \\ & - \frac{\{\boldsymbol{p}_x, b_x\} + \{\boldsymbol{p}_y, b_y\}}{2m}\boldsymbol{\tau}_3 - \Delta_0\left(b_x\,\boldsymbol{\tau}_1 + b_y\,\boldsymbol{\tau}_2\right) - b_0\,\boldsymbol{\tau}_0 - \frac{a_x^2 + a_y^2}{2m}\boldsymbol{\tau}_3 - \frac{-a_x b_x - a_y b_y}{m}\boldsymbol{\tau}_0 - \frac{b_x^2 + b_y^2}{2m}\boldsymbol{\tau}_3 \end{aligned} \tag{115}$$

where we defined the fields[5]

$$a_0 = A_0 - \partial_t\theta, \qquad \boldsymbol{a} = \mathbf{A} - \nabla\theta, \qquad b_t = \partial_t\gamma, \qquad \boldsymbol{b} = \nabla\gamma \tag{116}$$

G Showing that there is no coupling between the 3-vector fields *a* and *b*

We start by showing that all the integrands of the correlators $\Pi^{\mu\nu}_{ab}$ are antisymmetric with respect to the momentum-frequency (2+1) vector, k:

$$\mathrm{tr}\left(\mathcal{G}(k+\frac{q}{2})j^0_a(\boldsymbol{k})\mathcal{G}(k-\frac{q}{2})j^0_b(\boldsymbol{k})\right) = \mathrm{tr}\left(\frac{(-(k_0-\frac{q_0}{2})\boldsymbol{\tau}_0 - \boldsymbol{g}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})(-\boldsymbol{\tau}_3)(-(k_0+\frac{q_0}{2})\boldsymbol{\tau}_0 - \boldsymbol{g}_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_0}{\left(-(k_0+\frac{q_0}{2})^2 + g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2 + g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}\right) =$$
$$\frac{2i(g_{1,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{2,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}} - g_{2,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{1,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}) - 2((k_0-\frac{q_0}{2})g_{3,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}} + (k_0+\frac{q_0}{2})g_{3,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}})}{\left(-(k_0+\frac{q_0}{2})^2 + g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2 + g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)} =$$
$$-\,\mathrm{tr}\left(\mathcal{G}(-k+\frac{q}{2})j^0_a(-\boldsymbol{k})\mathcal{G}(-k-\frac{q}{2})j^0_b(-\boldsymbol{k})\right) \tag{117}$$

$$\mathrm{tr}\left(\mathcal{G}(k+\frac{q}{2})j^1_a(\boldsymbol{k})\mathcal{G}(k-\frac{q}{2})j^0_b(\boldsymbol{k})\right) = \mathrm{tr}\left(\frac{(-(k_0-\frac{q_0}{2})\boldsymbol{\tau}_0 - \boldsymbol{g}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})(-\partial_{k_1}g_{3,\boldsymbol{k}}\boldsymbol{\tau}_0)(-(k_0+\frac{q_0}{2})\boldsymbol{\tau}_0 - \boldsymbol{g}_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_0}{\left(-(k_0+\frac{q_0}{2})^2 + g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2 + g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}\right) =$$
$$\frac{-2\left((k_0-\frac{q_0}{2})(k_0+\frac{q_0}{2}) + \boldsymbol{g}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\boldsymbol{g}_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}{\left(-(k_0+\frac{q_0}{2})^2 + g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2 + g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}\partial_{k_1}g_{3,\boldsymbol{k}} = -\mathrm{tr}\left(\mathcal{G}(-k+\frac{q}{2})j^1_a(-\boldsymbol{k})\mathcal{G}(-k-\frac{q}{2})j^0_b(-\boldsymbol{k})\right) \tag{118}$$

$$\mathrm{tr}\left(\mathcal{G}(k+\frac{q}{2})j^2_a(\boldsymbol{k})\mathcal{G}(k-\frac{q}{2})j^0_b(\boldsymbol{k})\right) = \mathrm{tr}\left(\frac{(-(k_0-\frac{q_0}{2})\boldsymbol{\tau}_0 - \boldsymbol{g}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})(-\partial_{k_2}g_{3,\boldsymbol{k}}\boldsymbol{\tau}_0)(-(k_0+\frac{q_0}{2})\boldsymbol{\tau}_0 - \boldsymbol{g}_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})\boldsymbol{\tau}_0}{\left(-(k_0+\frac{q_0}{2})^2 + g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2 + g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}\right) =$$
$$\frac{-2\left((k_0-\frac{q_0}{2})(k_0+\frac{q_0}{2}) + \boldsymbol{g}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\boldsymbol{g}_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}{\left(-(k_0+\frac{q_0}{2})^2 + g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2 + g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}\partial_{k_2}g_{3,\boldsymbol{k}} = -\mathrm{tr}\left(\mathcal{G}(-k+\frac{q}{2})j^2_a(-\boldsymbol{k})\mathcal{G}(-k-\frac{q}{2})j^0_b(-\boldsymbol{k})\right) \tag{119}$$

[5]For $\Delta = \Delta_0 e^{i\theta(\mathbf{x},t)}/2$ the corresponding transformations are $\mathbf{U} = \lambda e^{-i\tau_z\theta(\boldsymbol{x},t)/2}$ and $\lambda = e^{-i\gamma(\boldsymbol{x},t)}$. In this case, the fields are defined as $a_0 = A_0 - \partial_t\theta/2$, $\boldsymbol{a} = \mathbf{A} - \nabla\theta/2$, $b_t = \partial_t\gamma$ and $\boldsymbol{b} = \nabla\gamma$.

$$
\begin{aligned}
&\mathrm{tr}\left(\mathcal{G}(k+\frac{q}{2})j_a^1(\boldsymbol{k})\mathcal{G}(k-\frac{q}{2})j_b^1(\boldsymbol{k})\right)= \\
&\qquad \mathrm{tr}\left(\frac{(-(k_0-\frac{q_0}{2})\boldsymbol{\tau}_0-\boldsymbol{g}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})(-\partial_{k_1}g_{3,\boldsymbol{k}}\boldsymbol{\tau}_0)(-(k_0+\frac{q_0}{2})\boldsymbol{\tau}_0-\boldsymbol{g}_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})\partial_{k_1}\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau}}{\left(-(k_0+\frac{q_0}{2})^2+g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2+g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}\right)= \\
&\frac{\left(\begin{array}{c} 2\left((k_0-\frac{q_0}{2})\boldsymbol{g}_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\cdot\partial_{k_1}\boldsymbol{g}_{\boldsymbol{k}}+(k_0+\frac{q_0}{2})\boldsymbol{g}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\cdot\partial_{k_1}\boldsymbol{g}_{\boldsymbol{k}}\right)+2i\left(g_{3,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}g_{2,\boldsymbol{k}}\partial_{k_1}g_{1,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right. \\ \left.-g_{2,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}g_{3,\boldsymbol{k}}\partial_{k_1}g_{1,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}-g_{1,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}g_{2,\boldsymbol{k}}\partial_{k_1}g_{3,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}+g_{2,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}g_{1,\boldsymbol{k}}\partial_{k_1}g_{3,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\end{array}\right)}{\left(-(k_0+\frac{q_0}{2})^2+g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2+g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}\partial_{k_1}g_{3,\boldsymbol{k}}= \\
&\qquad\qquad -\mathrm{tr}\left(\mathcal{G}(-k+\frac{q}{2})j_a^1(-\boldsymbol{k})\mathcal{G}(-k-\frac{q}{2})j_b^1(-\boldsymbol{k})\right)
\end{aligned} \tag{120}
$$

$$
\begin{aligned}
&\mathrm{tr}\left(\mathcal{G}(k+\frac{q}{2})j_a^2(\boldsymbol{k})\mathcal{G}(k-\frac{q}{2})j_b^1(\boldsymbol{k})\right)= \\
&\qquad \mathrm{tr}\left(\frac{(-(k_0-\frac{q_0}{2})\boldsymbol{\tau}_0-\boldsymbol{g}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})(-\partial_{k_2}g_{3,\boldsymbol{k}}\boldsymbol{\tau}_0)(-(k_0+\frac{q_0}{2})\boldsymbol{\tau}_0-\boldsymbol{g}_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})\partial_{k_1}\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau}}{\left(-(k_0+\frac{q_0}{2})^2+g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2+g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}\right)= \\
&\frac{\left(\begin{array}{c} -2\left((k_0-\frac{q_0}{2})\boldsymbol{g}_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\cdot\partial_{k_1}\boldsymbol{g}_{\boldsymbol{k}}+(k_0+\frac{q_0}{2})\boldsymbol{g}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\cdot\partial_{k_1}\boldsymbol{g}_{\boldsymbol{k}}\right)+2i\left(g_{3,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{2,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_1}g_{1,\boldsymbol{k}}\right. \\ \left.-g_{2,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{3,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_1}g_{1,\boldsymbol{k}}-g_{1,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{2,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_1}g_{3,\boldsymbol{k}}+g_{2,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{1,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_1}g_{3,\boldsymbol{k}}\right)\end{array}\right)}{\left(-(k_0+\frac{q_0}{2})^2+g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2+g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}\partial_{k_2}g_{3,\boldsymbol{k}}= \\
&\qquad\qquad -\mathrm{tr}\left(\mathcal{G}(-k+\frac{q}{2})j_a^2(-\boldsymbol{k})\mathcal{G}(-k-\frac{q}{2})j_b^1(-\boldsymbol{k})\right)
\end{aligned} \tag{121}
$$

$$
\begin{aligned}
&\mathrm{tr}\left(\mathcal{G}(k+\frac{q}{2})j_a^1(\boldsymbol{k})\mathcal{G}(k-\frac{q}{2})j_b^2(\boldsymbol{k})\right)= \\
&\qquad \mathrm{tr}\left(\frac{(-(k_0-\frac{q_0}{2})\boldsymbol{\tau}_0-\boldsymbol{g}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})(-\partial_{k_1}g_{3,\boldsymbol{k}}\boldsymbol{\tau}_0)(-(k_0+\frac{q_0}{2})\boldsymbol{\tau}_0-\boldsymbol{g}_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})\partial_{k_2}\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau}}{\left(-(k_0+\frac{q_0}{2})^2+g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2+g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}\right)= \\
&\frac{\left(\begin{array}{c} -2\left((k_0-\frac{q_0}{2})\boldsymbol{g}_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\cdot\partial_{k_2}\boldsymbol{g}_{\boldsymbol{k}}+(k_0+\frac{q_0}{2})\boldsymbol{g}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\cdot\partial_{k_2}\boldsymbol{g}_{\boldsymbol{k}}\right)+2i\left(g_{1,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{3,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_2}g_{2,\boldsymbol{k}}\right. \\ \left.-g_{3,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{1,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_2}g_{2,\boldsymbol{k}}-g_{1,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{2,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_2}g_{3,\boldsymbol{k}}+g_{2,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{1,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_2}g_{3,\boldsymbol{k}}\right)\end{array}\right)}{\left(-(k_0+\frac{q_0}{2})^2+g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2+g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}\partial_{k_1}g_{3,\boldsymbol{k}}= \\
&\qquad\qquad -\mathrm{tr}\left(\mathcal{G}(-k+\frac{q}{2})j_a^1(-\boldsymbol{k})\mathcal{G}(-k-\frac{q}{2})j_b^2(-\boldsymbol{k})\right)
\end{aligned} \tag{122}
$$

$$\operatorname{tr}\left(\mathcal{G}(k+\frac{q}{2})j_a^2(\boldsymbol{k})\mathcal{G}(k-\frac{q}{2})j_b^2(\boldsymbol{k})\right)=$$

$$\operatorname{tr}\left(\frac{(-(k_0-\frac{q_0}{2})\boldsymbol{\tau}_0-\boldsymbol{g}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})(-\partial_{k_2}g_{3,\boldsymbol{k}}\boldsymbol{\tau}_0)(-(k_0+\frac{q_0}{2})\boldsymbol{\tau}_0-\boldsymbol{g}_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})\partial_{k_2}\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau}}{\left(-(k_0+\frac{q_0}{2})^2+g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2+g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}\right)=$$

$$\frac{\begin{pmatrix}-2\left((k_0-\frac{q_0}{2})\boldsymbol{g}_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\cdot\partial_{k_2}\boldsymbol{g}_{\boldsymbol{k}}+(k_0+\frac{q_0}{2})\boldsymbol{g}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\cdot\partial_{k_2}\boldsymbol{g}_{\boldsymbol{k}}\right)+2i\left(g_{1,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{3,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_2}g_{2,\boldsymbol{k}}\right.\\ \left.-g_{3,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{1,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_2}g_{2,\boldsymbol{k}}-g_{1,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{2,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_2}g_{3,\boldsymbol{k}}+g_{2,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{1,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_2}g_{3,\boldsymbol{k}}\right)\end{pmatrix}}{\left(-(k_0+\frac{q_0}{2})^2+g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2+g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}\partial_{k_2}g_{3,\boldsymbol{k}}=$$

$$-\operatorname{tr}\left(\mathcal{G}(-k+\frac{q}{2})j_a^2(-\boldsymbol{k})\mathcal{G}(-k-\frac{q}{2})j_a^2(-\boldsymbol{k})\right)\quad(123)$$

$$\operatorname{tr}\left(\mathcal{G}(k+\frac{q}{2})j_a^0(\boldsymbol{k})\mathcal{G}(k-\frac{q}{2})j_b^1(\boldsymbol{k})\right)=$$

$$\operatorname{tr}\left(\frac{(-(k_0-\frac{q_0}{2})\boldsymbol{\tau}_0-\boldsymbol{g}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})(-\boldsymbol{\tau}_3)(-(k_0+\frac{q_0}{2})\boldsymbol{\tau}_0-\boldsymbol{g}_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})\partial_{k_1}\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau}}{\left(-(k_0+\frac{q_0}{2})^2+g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2+g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}\right)=$$

$$\frac{2\begin{pmatrix}-(k_0-\frac{q_0}{2})(k_0+\frac{q_0}{2})\partial_{k_1}g_{3,\boldsymbol{k}}+i(k_0-\frac{q_0}{2})g_{2,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_1}g_{1,\boldsymbol{k}}-ig_{2,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}(k_0+\frac{q_0}{2})\partial_{k_1}g_{1,\boldsymbol{k}}+g_{1,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{1,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_1}g_{3,\boldsymbol{k}}\\ +g_{2,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{2,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_1}g_{3,\boldsymbol{k}}-g_{3,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{3,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_1}g_{3,\boldsymbol{k}}-g_{3,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{1,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_1}g_{1,\boldsymbol{k}}-g_{1,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{3,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_1}g_{1,\boldsymbol{k}}\end{pmatrix}}{\left(-(k_0+\frac{q_0}{2})^2+g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2+g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}=$$

$$-\operatorname{tr}\left(\mathcal{G}(-k+\frac{q}{2})j_a^0(-\boldsymbol{k})\mathcal{G}(-k-\frac{q}{2})j_b^1(-\boldsymbol{k})\right)\quad(124)$$

$$\operatorname{tr}\left(\mathcal{G}(k+\frac{q}{2})j_a^0(\boldsymbol{k})\mathcal{G}(k-\frac{q}{2})j_b^2(\boldsymbol{k})\right)=$$

$$\operatorname{tr}\left(\frac{(-(k_0-\frac{q_0}{2})\boldsymbol{\tau}_0-\boldsymbol{g}_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})(-\boldsymbol{\tau}_3)(-(k_0+\frac{q_0}{2})\boldsymbol{\tau}_0-\boldsymbol{g}_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\cdot\boldsymbol{\tau})\partial_{k_2}\boldsymbol{g}_{\boldsymbol{k}}\cdot\boldsymbol{\tau}}{\left(-(k_0+\frac{q_0}{2})^2+g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2+g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}\right)=$$

$$\frac{2\begin{pmatrix}-(k_0-\frac{q_0}{2})(k_0+\frac{q_0}{2})\partial_{k_2}g_{3,\boldsymbol{k}}-i(k_0-\frac{q_0}{2})g_{1,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_2}g_{2,\boldsymbol{k}}+ig_{1,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}(k_0+\frac{q_0}{2})\partial_{k_2}g_{2,\boldsymbol{k}}+g_{1,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{1,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_2}g_{3,\boldsymbol{k}}\\ -g_{2,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{2,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_2}g_{3,\boldsymbol{k}}-g_{3,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{3,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_2}g_{3,\boldsymbol{k}}-g_{3,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{2,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_2}g_{2,\boldsymbol{k}}+g_{2,\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}g_{3,\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\partial_{k_2}g_{2,\boldsymbol{k}}\end{pmatrix}}{\left(-(k_0+\frac{q_0}{2})^2+g^2_{\boldsymbol{k}+\frac{\boldsymbol{q}}{2}}\right)\left(-(k_0-\frac{q_0}{2})^2+g^2_{\boldsymbol{k}-\frac{\boldsymbol{q}}{2}}\right)}=$$

$$-\operatorname{tr}\left(\mathcal{G}(-k+\frac{q}{2})j_a^0(-\boldsymbol{k})\mathcal{G}(-k-\frac{q}{2})j_b^2(-\boldsymbol{k})\right)\quad(125)$$

We shown above that the integrand of the correlator Π_{ab} is antisymmetry with respect to k, .i.e.

$$\operatorname{tr}\left(\mathcal{G}_0(k+\frac{q}{2})j_a(k)\mathcal{G}_0(k-\frac{q}{2})j_b(k)\right)=-\operatorname{tr}\left(\mathcal{G}_0(-k+\frac{q}{2})j_a(-k)\mathcal{G}_0(-k-\frac{q}{2})j_b(-k)\right).\quad(126)$$

The integrand of the correlator Π_{ba} is also anti-symmetric with respect to k. It can easily shown by (a) using the cyclic property of the trace operation to reverse the order of the fields in both sides and (b) replacing q with $-q$ in both sides to get correlators into the right form.

In conclusion: Since the correlators involve integration over the whole momentum-frequency space and the integrands are anti-symmetric, they are all equal zero. Thus, there is no coupling between the 3-vector fields a and b. This is a general property that holds to all orders of the gradient expansion.

H The vortices exchange statistics

The phase of the p-wave gap is defined as

$$\theta = \sum_{j(\neq i)} \theta_{ij} \tag{127}$$

Where $\theta_{ij} \equiv \arg\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right)$ is the phase that relates a particle at site, $\boldsymbol{r}_i$ with a vortex located at $\boldsymbol{r}_j$.

The argument function is defined as the angle between the x axis and the vector $\boldsymbol{r}$ and is given by

$$\arg(\boldsymbol{r}) = 2\arctan\left(\frac{y}{x - \sqrt{x^2 + y^2}}\right) + \pi(1 + 2\ell) \tag{128}$$

with ℓ being the branch number.

Applying a suitable gauge transformation, the phase of the superconducting order parameter, $\Delta(r) = \Delta_0 e^{i\theta(\boldsymbol{r})}$ is transmuted into a potential field, $\boldsymbol{a}(\boldsymbol{r}_i)$. Under this kind of gauge, adding vortices to the system through the order parameter is equivalent to placing magnetic flux of quanta Φ_0. One should emphasize that only singular gauge transformation create fluxiods.

The vector potential field $\boldsymbol{a}(\boldsymbol{r}_i)$, associated with phase θ by the gauge transformation

$$\mathbf{U} = e^{-i\tau_z\theta/2}, \tag{129}$$

is given by

$$\begin{aligned} \frac{2\pi}{\Phi_0}\boldsymbol{a}(\boldsymbol{r}_i) &= \sum_{j(\neq i)} \boldsymbol{\nabla}_{\boldsymbol{r}_i}\theta_{ij} = \sum_{j(\neq i)} \boldsymbol{\nabla}_{\boldsymbol{r}_i}\arctan\left(\frac{y_{ij}}{x_{ij}}\right) = \sum_{j(\neq i)} \frac{1}{1 + \frac{y_{ij}^2}{x_{ij}^2}}\left(\frac{1}{x_{ij}}\hat{\mathbf{y}} - \frac{y_{ij}}{x_{ij}^2}\hat{\mathbf{x}}\right) \\ &= \sum_{j(\neq i)} \frac{x_{ij}\hat{\mathbf{y}} - y_{ij}\hat{\mathbf{x}}}{x_{ij}^2 + y_{ij}^2} = \sum_{j(\neq i)} \frac{\hat{\mathbf{z}} \times \hat{\boldsymbol{r}}_{ij}}{r_{ij}} \end{aligned} \tag{130}$$

Where $\boldsymbol{r}_{ij}$ defined as $\boldsymbol{r}_j - \boldsymbol{r}_i$ and $\Phi_0 = \pi$ in natural units ($\Phi_0 = h/2e$ in SI units). It will be shown as useful to define the potential at $\boldsymbol{r}_i$ due to a vortex that located at $\boldsymbol{r}_j$ by

$$\boldsymbol{a}_{ij} \equiv \frac{\Phi_0}{2\pi}\frac{\hat{\mathbf{z}} \times \hat{\boldsymbol{r}}_{ij}}{r_{ij}}. \tag{131}$$

From hereby we regard the factor $\Phi_0/2\pi$ as unity and recover it only in the end of the calculation. Since the gauge transformation is singular, the vector potential creates a perpendicular magnetic field (The terms charge, magnetic field and flux, in this context, are borrowed from the terminology of the Maxwell's electrodynamics) of strength

$$\left(\boldsymbol{\nabla}_{\boldsymbol{r}_i} \times \boldsymbol{a}\right) \cdot \hat{\mathbf{z}} = \sum_{j(\neq i)} \begin{cases} \left(\boldsymbol{\nabla}_{\boldsymbol{r}_i} \times \boldsymbol{a}_{ij}\right) \cdot \hat{\mathbf{z}} = \left(\boldsymbol{\nabla}_{\boldsymbol{r}_i} \times \boldsymbol{\nabla}_{\boldsymbol{r}_i}\theta\right) \cdot \hat{\mathbf{z}} \;=\; 0 & , \boldsymbol{r}_{ij} \neq 0 \\ \\ \lim\limits_{\boldsymbol{r}_{ij}\to 0} \frac{1}{\pi r_{ij}^2} \int\limits_0^{2\pi} \left(\frac{1}{r_{ij}}\hat{\boldsymbol{\varphi}}_{ij}\right) \cdot \left(\hat{\boldsymbol{\varphi}}_{ij} r_{ij} d\varphi_{ij}\right) = \lim\limits_{r_{ij}\to 0} \frac{2}{r_{ij}^2} \;=\; \infty & , \boldsymbol{r}_{ij} \;=\; 0 \end{cases} \tag{132}$$

The result of the first case $\boldsymbol{r}_{ij} \neq 0$ is trivial. A curl over a gradient of a scalar field in a simply connected subregion is always zero. This result can easily obtained by using the deferential form of the curl operator over the field $\boldsymbol{a}$. In the second case $\boldsymbol{r}_{ij} = 0$, the derivatives at this point are not well-defined and the curl is calculated directly from its definition

$$(\nabla \times \boldsymbol{a}) \cdot \hat{\mathbf{n}} \equiv \lim_{S\to 0}\left(\frac{1}{|S|}\oint_\Gamma \boldsymbol{a} \cdot d\mathbf{l}\right) \tag{133}$$

Here, $\oint_\Gamma \boldsymbol{a} \cdot d\mathbf{l}$ is a line integral along the boundary of the area in question, $|S|$ is the magnitude of the area, $\hat{\mathbf{n}}$ is the unit vector perpendicular to the plane and $d\mathbf{l}$ is tangent to Γ and pointing anticlockwise with respect to $\hat{\mathbf{n}}$. We apply the linear operator, curl on each term in the sum separately. Since the calculation should hold for any arbitrary closed contour we can choose a different loop, that would ease the integration, for each term. The contour of the term, $\boldsymbol{a}_{ij}$ is chosen to be a circle centred at $\boldsymbol{r}_j$.

The result of the curl above seems proportional to a delta function. In order to find its proportionality coefficient we use Kelvin–Stokes theorem,

$$\oint_\Gamma \mathbf{a}\, d\mathbf{l} = \iint_S \nabla \times \mathbf{a}\, d\mathbf{s}. \tag{134}$$

Where the notations and integration contours are the same as used to calculate the curl by its definition. $d\mathbf{s}$ is perpendicular to the plane enclosed by the path integral and positively oriented. Calculating explicitly the l.h.s of the theorem yields

$$\sum_{j(\neq i)} 2\pi = \int d^2\boldsymbol{r}\; \boldsymbol{\nabla} \times \boldsymbol{a}. \tag{135}$$

from which we can deduce that

$$\boldsymbol{\nabla} \times \boldsymbol{a} = 2\pi \sum_{j(\neq i)} \delta(\boldsymbol{r}_{ij})\, \hat{\mathbf{z}}. \tag{136}$$

Introducing back the factor $\Phi_0/2\pi$, the relevant expressions to the integration over the Chern-Simons term are

$$\nabla \times \boldsymbol{a} = \Phi_0 \sum_{j(\neq i)} \delta(\boldsymbol{r}_{ij})\, \hat{\mathbf{z}}. \tag{137}$$

where

$$\boldsymbol{a}\,(\boldsymbol{r}_i) = \frac{\Phi_0}{2\pi} \sum_{j(\neq i)} \frac{\hat{\boldsymbol{\varphi}}_{ij}}{r_{ij}} \tag{138}$$

and the factor Φ_0 in the expression reflects the fact that these are single-quantum vortex in a p-wave superconductor of spinless fermions.

The expression for the scalar potential $-a_0$, the time derivative of the phase $\theta/2$, is

$$\frac{2\pi}{\Phi_0} a_0 = \partial_t \theta = \sum_{j(\neq i)} \partial_t \theta_{ij} = \sum_{j(\neq i)} \frac{y_{ij}\dot{x}_j - x_{ij}\dot{y}_j}{x_{ij}^2 + y_{ij}^2} \tag{139}$$

Usually, the particles and fluxoids move quite independently of each other. However, if the transformation introduce a Chern-Simons term,

$$L_{cs} = \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda, \tag{140}$$

into the Lagrangian then the particles are endows with the flux. Thus, when one of our particles move around another, the effective action acquires a phase. We demonstrate the exchange process by calculating the contribution of the Chern-Simons term in the case of the exchange of two particles. The path of the interchange between two vortices is illustrated in Figure 3(c). Integrating the Chern-Simons term for the interchange yields

$$\begin{aligned}
\int d^2 r_i dt\, L_{cs} \quad = \quad & \int d^2 r_i dt\, \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda = \int d^2 r_i dt\; a_0(\partial_1 a_2 - \partial_2 a_1) = \\
& \int d^2 r_i dt \left(\frac{\Phi_0}{2\pi} \sum_{j(\neq i)} \partial_t \theta_{ij}\right)(\nabla \times \boldsymbol{a})_{\mathbf{z}} = \int d^2 r_i dt\; \left(\frac{\Phi_0}{2\pi} \sum_{j(\neq i)} \partial_t \theta_{ij}\right)\left(\Phi_0 \sum_{j(\neq i)} \delta(\boldsymbol{r}_{ij})\right) = \\
& -\frac{\Phi_0^2}{2\pi} \int d^2 r_i dt\; \sum_{j(\neq i)} \frac{x_{ij}\dot{y}_j - y_{ij}\dot{x}_j}{x_{ij}^2 + y_{ij}^2}\,(\delta(\boldsymbol{r}_i - \boldsymbol{r}_1) + \delta(\boldsymbol{r}_i - \boldsymbol{r}_2)) = \\
& -\frac{\Phi_0^2}{2\pi} \int dt\; \left(\frac{x_{12}\dot{y}_2 - y_{12}\dot{x}_2}{x_{12}^2 + y_{12}^2} + \frac{x_{21}\dot{y}_1 - y_{21}\dot{x}_1}{x_{21}^2 + y_{21}^2}\right) = \frac{\Phi_0^2}{2\pi} \int_{t_i}^{t_f} dt\; \frac{x_{12}\dot{y}_{12} - y_{12}\dot{x}_{12}}{x_{12}^2 + y_{12}^2} = \\
& \frac{\Phi_0^2}{2\pi} \int_{\theta(t_i)=0}^{\theta(t_f)=\pi} d\theta_{12} = \frac{\Phi_0^2}{2}
\end{aligned}$$

where in the last step we switched to relative coordinates defined as

$$\boldsymbol{r}_{21} = \boldsymbol{r}_2 - \boldsymbol{r}_1, \quad \boldsymbol{\rho} = \boldsymbol{r}_2 + \boldsymbol{r}_1. \tag{141}$$

The term $\hat{\mathbf{z}} \times \boldsymbol{r}_i j \cdot \dot{\boldsymbol{r}}_{ij}$ equals zero when $\dot{\boldsymbol{r}}_{ij}$ is parallel to $\boldsymbol{r}_{ij}$. Thus, radial parts of the contour do not contribute to the integration. Hence, the two operations, exchange of two particles and taking one particle half a rotation around the other, are equivalent. To ease the integration we take the contour as shown in Figure 3(c).

Next, we calculate the contribution of the CS like term due the presence of an external magnetic field $\mathbf{B}(\boldsymbol{r}) = \frac{\Phi_0}{2\pi\lambda^2} K_0\left(\frac{r}{\lambda}\right)\hat{\mathbf{z}}$ associated with a single-quantum vortex flux of Φ_0 [5]. The integration contour is taken as shown in Figure 3(b) and calculation is done in two steps. First, we we move particle 2 along the arc with radius R, while keeping particle 1 static. Then, we move both particles simultaneously in such way that separation distance between the two particles is kept constant. For the first part we get

$$\frac{1}{8\pi}\int d^2r_i dt\; a_0\mathbf{B}_z = \frac{1}{8\pi}\int d^2r_i dt\; \left(-\frac{\Phi_0}{2\pi}\partial_t\theta_{i2}\right)\left(\frac{\Phi_0}{2\pi\lambda^2}K_0\left(\frac{r_{i1}}{\lambda}\right)\right) = -\frac{\Phi_0^2}{16\pi\lambda^2}\int_0^R dr\; rK_0\left(\frac{r}{\lambda}\right) = \\ -\frac{\Phi_0^2}{16\pi}\left(1-\frac{R}{\lambda}K_1\left(\frac{R}{\lambda}\right)\right) = -\frac{\pi}{16}\left(1-\frac{R}{\lambda}K_1\left(\frac{R}{\lambda}\right)\right) \underset{R\longrightarrow\infty}{=} -\frac{\pi}{16} \quad (142)$$

Here λ is the penetration depth and we made the replacement $\theta_{i2}(t_f) - \theta_{i2}(t_i) \longrightarrow \pi\mathcal{H}(r_i - R)$. The justification of the replacement lies in the fact that for a given $\boldsymbol{r}_2 = (x_2, y_2)$ we can always map to each vector in the upper plane,$\boldsymbol{r}_+ = (x, y > 0)$ a vector in the lower plane, $\boldsymbol{r}_- = (\frac{x(y+y_2)-2x_2y_2}{y-y_2}, -y)$ such that $\arg(\boldsymbol{r}_+ - \boldsymbol{r}_2(t)) + \arg(\boldsymbol{r}_- - \boldsymbol{r}_2(t)) = 2\pi\Delta\ell$. Here $\Delta\ell$ represents the difference between the branches of the two arguments. In the beginning of the circulation, at time t_i, $\Delta\ell = 0$ but after particle 1 finished half a rotation, at time t_f, $\Delta\ell = \mathcal{H}(r - R)$.
For the second part we get

$$\frac{1}{8\pi}\int d^2r_i dt\; a_0\mathbf{B}_z = -\frac{\Phi_0^2}{32\pi^3\lambda^2}\int d^2r_i dt\; \left(\partial_t\theta_{i2}K_0\left(\frac{r_{i1}}{\lambda}\right) + \partial_t\theta_1 K_0\left(\frac{r_{i2}}{\lambda}\right)\right) = \\ -\frac{\Phi_0^2}{32\pi^3\lambda^2}\int d^2rdt\; \left(\partial_t\arg(\boldsymbol{r}-\boldsymbol{r}_2)K_0\left(\frac{|\boldsymbol{r}-\boldsymbol{r}_1|}{\lambda}\right) + \partial_t\arg(\boldsymbol{r}_i-\boldsymbol{r}_1)K_0\left(\frac{|\boldsymbol{r}-\boldsymbol{r}_2|}{\lambda}\right)\right) = \\ -\frac{\Phi_0^2}{32\pi^3\lambda^2}\int d^2rdt\; \left(\partial_t\arg(\boldsymbol{r}-\boldsymbol{r}_2)K_0\left(\frac{|\boldsymbol{r}-\boldsymbol{r}_2-\mathbf{R}|}{\lambda}\right) + \partial_t\arg(\boldsymbol{r}-\boldsymbol{r}_2-\mathbf{R})K_0\left(\frac{|\boldsymbol{r}-\boldsymbol{r}_2|}{\lambda}\right)\right) = \\ -\frac{\Phi_0^2}{32\pi^3\lambda^2}\int d^2rdt\; \left[\partial_t\arg(\boldsymbol{r}-\boldsymbol{r}_2+\mathbf{R}) + \partial_t\arg(\boldsymbol{r}-\boldsymbol{r}_2-\mathbf{R})\right]K_0\left(\frac{|\boldsymbol{r}-\boldsymbol{r}_2|}{\lambda}\right) = 0 \quad (143)$$

Here $\boldsymbol{r}_2(t) - \boldsymbol{r}_1(t) = \mathbf{R}$, $\mathbf{R} = (R, 0)$ and $\boldsymbol{r}_2(t) = (x_2(t), 0)$. In order to show that this part of the integration do not contribute, we notice can always map to each vector in the upper plane,$\boldsymbol{r}_+ = (x, y > 0)$ a vector in the lower plane, $\boldsymbol{r}_- = (x, -y)$ such that $\arg(\boldsymbol{r}_+ - \boldsymbol{r}_2(t) \pm \mathbf{R}) + \arg(\boldsymbol{r}_- - \boldsymbol{r}_2(t) \pm \mathbf{R}) = 0$.

We found that the contribution of the external magnetic field cancels exactly the contribution from the collective response of the condensate when the distance between the two vortices is infinite long, $R \longrightarrow \infty$. This demonstrates that when the exchange is performed with large vortex separations compared to λ, this CS-like term does not contribute to the exchange phase. At small distances, non-universal contributions will occur.

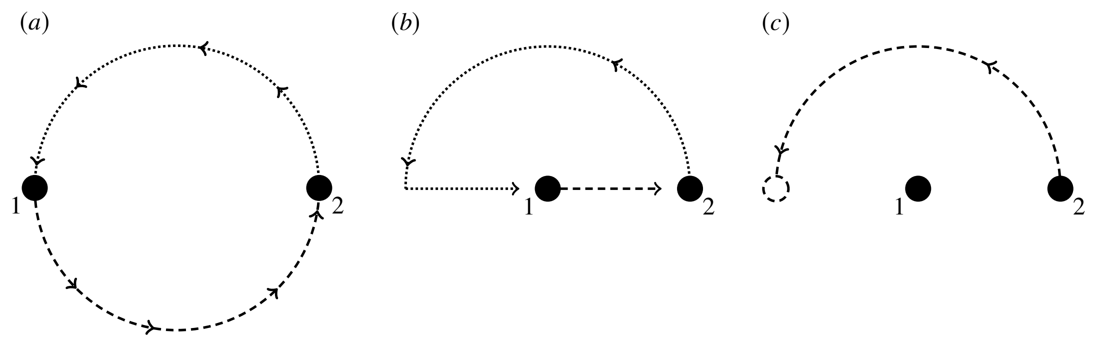

Figure 3: Equivalent contours of integration for the interchange of two particles. For the deficient CS term, The equivalence is a consequence of the integral being linear to the arc central angle, while radial parts do not contribute. For the complete CS term, the equivalence is due to the integral being linear to the winding number.

The second addition to the vector potential, $\boldsymbol{b}(\boldsymbol{r}_i)$ due to the gauge transformation

$$\mathbf{U} = e^{-i\gamma(\boldsymbol{r}_i)} \tag{144}$$

where

$$\gamma(\boldsymbol{r}_i) = \pi \sum_{j(\neq i)} \underbrace{\left(\sum_{\ell=1}^{\infty} \mathcal{H}(\theta_{ij} - 2\pi\ell) - \sum_{\ell=0}^{\infty} \mathcal{H}(-2\pi\ell - \theta_{ij}) \right)}_{\tilde{\mathcal{H}}_{ij}} \tag{145}$$

is given by

$$\frac{\pi}{\Phi_0}\boldsymbol{b} = \boldsymbol{\nabla}_i \gamma = \pi \sum_{j(\neq i)} \left(\sum_{\ell=1}^{\infty} \delta(\theta_{ij} - 2\pi\ell) - \sum_{\ell=0}^{\infty} \delta(-2\pi\ell - \theta_{ij}) \right) \boldsymbol{\nabla}_i \theta_{ij} = \pi \sum_{j(\neq i)} \underbrace{\sum_{\ell=-\infty}^{\infty} \delta(\theta_{ij} - 2\pi\ell)}_{\tilde{\delta}(\theta_{ij})} \boldsymbol{\nabla}_i \theta_{ij}. \tag{146}$$

The curl of the vector field $\boldsymbol{b}$ can be inferred from the Kelvin-Stokes theorem. The contour integration for the j term in the sum is given by

$$\frac{\pi}{\Phi_0} \int_{\Gamma} \boldsymbol{b}_{ij} d\mathbf{l} = \pi \int_0^{2\pi} (\delta(\theta_{ij}) + \delta(\theta_{ij} - 2\pi)) \frac{\hat{\theta}_{ij}}{r_{ij}} (\hat{\theta}_{ij} r_{ij} d\theta_{ij}) = \begin{cases} \pi, & j \text{ is inside } \Gamma \\ 0, & j \text{ is outside } \Gamma \end{cases} \tag{147}$$

Where the notations and contour are the same as used to calculate $\boldsymbol{\nabla} \times \boldsymbol{a}$ in Eq.(132). The result does not depend on the specific shape on the closed contour, that is to say that if the point $\boldsymbol{r}_j$ is inside the closed loop the integration would yield Φ_0 and otherwise zero. Thus, the curl of $\boldsymbol{b}$ must fulfil

$$\Phi_0 = \int_S \nabla \times \mathbf{b}_{ij} \, d\mathbf{s}. \tag{148}$$

Thus, we deduce that

$$\boldsymbol{\nabla} \times \boldsymbol{b} = \Phi_0 \sum_{j(\neq i)} \delta(\boldsymbol{r}_{ij}) \hat{\mathbf{z}} \tag{149}$$

Also, the expression for the field b_0, the time derivative of $\gamma(\boldsymbol{r}_i)$, is given by

$$\frac{\pi}{\Phi_0} b_0 = \partial_t \gamma = -\pi \sum_{j(\neq i)} \sum_{\ell=-\infty}^{\infty} \delta(\theta_{ij} - 2\pi\ell) \boldsymbol{\nabla}_j \theta_{ij} \dot{\boldsymbol{r}}_j = -\pi \sum_{j(\neq i)} \tilde{\delta}(\theta_{ij}) \frac{\hat{\mathbf{z}} \times \hat{\boldsymbol{r}}_{ij}}{r_{ij}} \dot{\boldsymbol{r}}_j \tag{150}$$

In order to stress the fields special properties, we write the field in Cartesian coordinates. Under this coordinate system, the Dirac delta is

$$\sum_{\ell} \delta(\theta_{ij} - 2\pi\ell) = x_{ij} \mathcal{H}(x_{ij}) \delta(y_{ij}) \tag{151}$$

so the field $\boldsymbol{b}$ can be written as

$$\boldsymbol{b}_{ij} = \Phi_0 \mathcal{H}(x_{ij}) \delta(y_{ij}) \hat{\mathbf{y}}. \tag{152}$$

and the expression for the field b_0 is

$$b_0 = \Phi_0 \sum_{j(\neq i)} \mathcal{H}(x_{ij})\delta(y_{ij})\dot{y}_j \tag{153}$$

This means that result depends only on the number of times and the direction in which the contour crosses the line $y_{ij} = 0$.

The key to write the Dirac delta in terms of Cartesian coordinates is to notice that for a single vortex at the origin we have $\theta = 0 \pmod{2\pi}$ when $y = 0,\ x > 0$. We assume that in the vicinity of the positive axis $(x > 0, y = 0)$ we can always write the argument as $\theta[x(y), y]$ with $x(y)$ being some parameterization that depends on our specific problem. Thus, the total derivative of θ at the point $(x > 0, y = 0)$ is

$$\left.\frac{d\theta}{dy}\right|_{(x>0,y=0)} = \left.\left[\frac{\partial\theta}{\partial y} + \frac{\partial\theta}{\partial x}\frac{\partial x}{\partial y}\right]\right|_{(x>0,y=0)} = \left.\frac{x - y\partial_y x}{x^2 + y^2}\right|_{(x>0,y=0)} = \frac{1}{x} \tag{154}$$

and by composition rule of a one dimensional Dirac delta function we get

$$\sum_{\ell} \delta(\theta - 2\pi\ell) = \delta(y)/|d\theta/dy|_{(x>0,y=0)} = x\mathcal{H}(x)\delta(y) \tag{155}$$

We processed by calculating the contribution to the action due to the Chern-Simons term associated with the field $\boldsymbol{b}$ for the interchange two vortices. For the case that $\mu = 0$ we have

$$\begin{aligned}
\int d^2\mathbf{r_i}dt\, \epsilon^{0\nu\lambda} b_0\partial_\nu b_\lambda = \int d^2\boldsymbol{r}_i dt\ b_0(\partial_1 b_2 - \partial_2 b_1) = \\
\int d^2\boldsymbol{r}_i dt \left(\Phi_0 \sum_{j(\neq i)} \left(\sum_{\ell=-\infty}^{\infty} \delta(\theta_{ij} - 2\pi\ell)\right) \boldsymbol{\nabla}_j\theta_{ij} \cdot \dot{\boldsymbol{r}}_j\right)(\boldsymbol{\nabla} \times \boldsymbol{b})_{\mathbf{z}} = \\
\int d^2\boldsymbol{r}_i dt \left(\Phi_0 \sum_{j(\neq i)} \tilde{\delta}(\theta_{ij})\boldsymbol{\nabla}_j\theta_{ij} \cdot \dot{\boldsymbol{r}}_j\right)\left(\Phi_0 \sum_{j(\neq i)} \delta(\boldsymbol{r}_{ij})\right) = \\
\Phi_0^2 \int dt\, \left(\tilde{\delta}(\theta_{12})\boldsymbol{\nabla}_2\theta_{12} \cdot \dot{\boldsymbol{r}}_2 + \tilde{\delta}(\theta_{21})\boldsymbol{\nabla}_1\theta_{21} \cdot \dot{\boldsymbol{r}}_1\right)
\end{aligned}$$

The calculation for $\mu \neq 0$ is more complicated and goes as follows:

$$\int d^2\mathbf{r_i}dt(1-\delta_{\mu 0})\epsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda = \int d^2\boldsymbol{r}_i dt\,(b_1\partial_2 b_0 - b_1\partial_0 b_2 + b_2\partial_0 b_1 - b_2\partial_1 b_0) =$$
$$\int d^2\boldsymbol{r}_i dt \sum_{j,k\neq j}\Big[b_1^{ij}\partial_2(b_1^{ik}\dot{x}_k + b_2^{ik}\dot{y}_k) - b_2^{ij}\partial_1(b_1^{ik}\dot{x}_k + b_2^{ik}\dot{y}_k) - b_1^{ij}(\partial_1 b_2^{ik}\dot{x}_k + \partial_2 b_2^{ik}\dot{y}_k)$$
$$+b_2^{ij}(\partial_1 b_1^{ik}\dot{x}^k + \partial_2 b_1^{ik}\dot{y}^k)\Big] = \int d^2\boldsymbol{r}_i dt \sum_{j,k\neq j}\underbrace{\left(b_1^{ij}\dot{x}_k + b_2^{ij}\dot{y}_k\right)}_{b_0^{ij}}\left(\partial_2 b_1^{ik} - \partial_1 b_2^{ik}\right) =$$
$$\int d^2\boldsymbol{r}_i dt\left[\left(b_1^{i1}\dot{x}_2 + b_2^{i1}\dot{y}_2\right)\left(\partial_2 b_1^{i2} - \partial_1 b_2^{i2}\right) + \left(b_1^{i2}\dot{x}^1 + b_2^{i2}\dot{y}_1\right)\left(\partial_2 b_1^{i1} - \partial_1 b_2^{i1}\right)\right] =$$
$$\Phi_0\int dt\left[\left(b_1^{21}\dot{x}_2 + b_2^{21}\dot{y}_2\right) + \left(b_1^{12}\dot{x}_1 + b_2^{12}\dot{y}_1\right)\right] = \Phi_0^2\int dt\left(\tilde{\delta}(\theta_{12})\boldsymbol{\nabla}_2\theta_{12}\cdot\dot{\boldsymbol{r}}_2 + \tilde{\delta}(\theta_{21})\boldsymbol{\nabla}_1\theta_{21}\cdot\dot{\boldsymbol{r}}_1\right) \quad (156)$$

We continue by switching to the relative coordinates which are defined as

$$\boldsymbol{r}_{21} = \boldsymbol{r}_2 - \boldsymbol{r}_1, \quad \boldsymbol{\rho} = \boldsymbol{r}_2 + \boldsymbol{r}_1 \quad (157)$$

$$\int d^2\boldsymbol{r}_i dt\,\epsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda = \Phi_0^2\int dt\left(\tilde{\delta}(\theta_{12})\boldsymbol{\nabla}_2\theta_{12}\cdot\dot{\boldsymbol{r}}_2 + \tilde{\delta}(\theta_{21})\boldsymbol{\nabla}_1\theta_{21}\cdot\dot{\boldsymbol{r}}_1\right) =$$
$$\Phi_0^2\int dt\left(\tilde{\delta}(\theta_{12})\boldsymbol{\nabla}_2\theta_{12} + \tilde{\delta}(\theta_{21})\boldsymbol{\nabla}_1\theta_{21}\right)\dot{\boldsymbol{\rho}} + \Phi_0^2\int dt\left(\tilde{\delta}(\theta_{12})\boldsymbol{\nabla}_2\theta_{12} - \tilde{\delta}(\theta_{21})\boldsymbol{\nabla}_1\theta_{21}\right)\dot{\boldsymbol{r}}_{21} =$$
$$\Phi_0^2\int dt\left(\tilde{\delta}(\theta_{21}-\pi) + \tilde{\delta}(\theta_{21})\right)\boldsymbol{\nabla}_{21}\theta_{21}\cdot\dot{\boldsymbol{r}}_{21} = \Phi_0^2\int_{\theta_{21}(t_i)=0}^{\theta_{21}(t_f)=\pi}\left(\tilde{\delta}(\theta_{21}-\pi) + \tilde{\delta}(\theta_{21})\right)d\theta_{21} = \Phi_0^2 \quad (158)$$

The exchange of the vortices is done simultaneously and with the same angular velocity as shown in Figure 3(a). Thus, $\boldsymbol{\rho} = 0$ along the interchange process. However, this selection is a matter of convenient since the exchange process depends only on the number of times branch cuts have been crossed and not on a specific contour. The following identities were used in the calculation above

$$\theta_{12} = \theta_{21} + \pi, \quad \boldsymbol{\nabla}_2\theta_{12} = \boldsymbol{\nabla}_{21}\theta_{21}, \quad \boldsymbol{\nabla}_1\theta_{21} = -\boldsymbol{\nabla}_{21}\theta_{21}. \quad (159)$$

I The density particle and current associated with the field $\boldsymbol{a}$

For the effective action of a 2D spinless p-wave superconductor,

$$\mathcal{S}(\mathbf{x},t) = \int \tfrac{d\boldsymbol{x}}{dt}\left(n\left(a_0 - \frac{1}{2m}\boldsymbol{a}^2\right) + \frac{m}{4\pi}a_0^2 - \frac{\kappa_a}{8\pi}\epsilon_{0jk}a_0\partial_j a_k + \frac{\kappa_b}{8\pi}\epsilon_{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda\right) \quad (160)$$

Where $\boldsymbol{a} = \mathbf{A} - \partial_{\boldsymbol{x}}\theta/2$, $a_0 = A_0 - \partial_t\theta/2$, $\boldsymbol{b} = \partial_{\boldsymbol{x}}\gamma$, $b_t = \partial_t\gamma$, $\kappa_a = \left(1 + \frac{2\mu}{m\Delta^2}H(-\mu)\right)^{-1}$, $\kappa_b = H(\mu)$ and $\Delta_0 = 2|\Delta|$.

In what follows we consider only the continuum limit with $\mu > 0$ for which the effective action is simply

$$\mathcal{S}(\mathbf{x}, t) = \int \frac{dx}{dt} \left[n\left(a_0 - \frac{\boldsymbol{a}^2}{2m}\right) + \frac{m}{4\pi}a_0^2 - \frac{1}{8\pi}a_0(\boldsymbol{\nabla}\times\boldsymbol{a})_z + \frac{1}{4\pi}b_0(\boldsymbol{\nabla}\times\boldsymbol{b})_z\right] \tag{161}$$

where we used the relation $\frac{1}{8\pi}\int \frac{dx}{dt}\epsilon_{\lambda\mu\nu}b_\lambda\partial_\mu b_\nu = \frac{1}{4\pi}\int \frac{dx}{dt}b_0(\boldsymbol{\nabla}\times\boldsymbol{b})_z$ that is proved in Appendix H.

The expression for particle density is

$$\rho = \frac{\delta S}{\delta a_0} = \frac{m}{2\pi}a_0 - \frac{1}{8\pi}(\boldsymbol{\nabla}\times\boldsymbol{a})_z + n \tag{162}$$

and for current density it is

$$j = \boldsymbol{\nabla}_{\boldsymbol{a}}\mathcal{S} = -\frac{n}{m}\boldsymbol{a}\boldsymbol{\nabla}_{\boldsymbol{a}}\boldsymbol{a} - \frac{1}{8\pi}a_0\boldsymbol{\nabla}_{\boldsymbol{a}}(\boldsymbol{\nabla}\times\boldsymbol{a})_z \tag{163}$$

By differentiation in parts, we write the term with curl operator as

$$a_0(\boldsymbol{\nabla}\times\boldsymbol{a})_z = \partial_x(a_0a_y) - a_y\partial_x a_0 - \partial_y(a_0a_x) + a_x\partial_y a_0 \tag{164}$$

The full spatial derivatives of fields that are bounded to be zero at infinity do not contribute to the effective action and can be neglected so the current can be written as

$$j = \boldsymbol{\nabla}_{\boldsymbol{a}}\mathcal{S} = -\frac{n}{m}\boldsymbol{a}\boldsymbol{\nabla}_{\boldsymbol{a}}\boldsymbol{a} + \frac{1}{8\pi}\boldsymbol{\nabla}_{\boldsymbol{a}}(a_x\partial_y a_0 - a_y\partial_x a_0) = -\frac{n}{m}\boldsymbol{a} - \frac{1}{8\pi}(\hat{\mathbf{z}}\times\boldsymbol{\nabla})a_0 \tag{165}$$

The vortex density in the weak pairing regime is proportional to

$$\rho_v = \partial_{b_0}S = \frac{1}{4\pi}\left(\partial_x b_y - \partial_y b_x\right) = \frac{1}{4}\sum_j \delta(\mathbf{r} - \mathbf{r}_j) \tag{166}$$

where $\mathbf{r}_j$ are the vortices coordinates.

J Magnus Force and the vortex mass

The Magnus force is a Lorentz-like force that acts on the vortex in the presence of a finite superfluid density. Starting from the term in the action

$$\int dt \iint d^2r\, n\partial_t\theta/2, \tag{167}$$

we write $\theta = \arg(\boldsymbol{r} - \mathbf{R}(t))$, where $\mathbf{R}(t)$ is the coordinate of the vortex. We can now write

$$\begin{aligned}\frac{n}{2}\int dt \iint d^2r\ \partial_t \arg(\boldsymbol{r} - \mathbf{R}(t)) \\ = -\frac{n}{2}\int dt \iint d^2r\ \partial_t \mathbf{R}\cdot \boldsymbol{\nabla}_{\mathbf{R}} \arg(\boldsymbol{r} - \mathbf{R}(t)) \\ = \int dt\ \dot{\mathbf{R}}\cdot\left[-\frac{n}{2}\iint d^2r\ \boldsymbol{\nabla}_{\mathbf{R}} \arg(\boldsymbol{r} - \mathbf{R}(t))\right]\end{aligned} \tag{168}$$

We define

$$\mathbf{A} = -\frac{n}{2}\int d^2r\ \boldsymbol{\nabla}_{\mathbf{R}} \arg(\boldsymbol{r} - \mathbf{R}(t)) \tag{169}$$

with associated "magnetic field" $\mathbf{B}$

$$\mathbf{B} = \nabla_{\mathbf{R}} \times \mathbf{A} = \frac{n}{2}\int d^2r \boldsymbol{\nabla}_{\mathbf{R}} \times \boldsymbol{\nabla}_{\mathbf{R}} \arg(\boldsymbol{r} - \mathbf{R}(t)) = \pi n \hat{\mathbf{z}} \tag{170}$$

The vortex thus experiences a Lorentz force with associated magnetic field πn, where n is the superfluid density.

Also, we can generate an expression for the vortex mass by rewriting the second term in the action as $\int dt \frac{1}{2} m_v \dot{\mathbf{x}}^2$, where $m_v = \iint d^2r\ \frac{m}{4\pi}(\mathbf{A} - \boldsymbol{\nabla}_{\boldsymbol{r}}\theta/2)^2$.

K Square lattice

The Hamiltonian density for the case that no electromagnetic fields penetrate the superconductor, there are no vortices and the coupling constant Δ is real (this can always be accomplished by the gauge transformation $e^{i\tau_3\theta/2}$, where θ is the phase of the order parameter) [6] is

$$\mathcal{H} = \left(\frac{\boldsymbol{p}^2}{2m} - \mu\right)\boldsymbol{\tau}_3 + 2\Delta \boldsymbol{p}_x \boldsymbol{\tau}_1 + 2\Delta \boldsymbol{p}_y \boldsymbol{\tau}_2. \tag{171}$$

Using the approximation $(1 - \cos(\boldsymbol{p}_x a)) \approx (\boldsymbol{p}_x a)^2/2$ and $\sin(\boldsymbol{p}_x a)/a \approx \boldsymbol{p}_x$ and the corresponding approximation for the y component, we find that the Hamiltonian density for a square lattice is

$$H = -\left[\frac{1}{m}(\cos \boldsymbol{p}_x + \cos \boldsymbol{p}_y) + \left(\mu - \frac{2}{m}\right)\right]\boldsymbol{\tau}_3 + 2\Delta \sin \boldsymbol{p}_x \boldsymbol{\tau}_1 + 2\Delta \sin \boldsymbol{p}_y \boldsymbol{\tau}_2 \tag{172}$$

where a is the lattice constant and from hereby we take it to be unity. The eigenenergies are

$$E_{\boldsymbol{k}} = \pm\sqrt{\left[\frac{1}{m}(\cos k_x + \cos k_y) + \left(\mu - \frac{2}{m}\right)\right]^2 + 4\Delta^2\left(\sin^2 k_x + \sin^2 k_y\right)} \tag{173}$$

[6] Δ is the same order parameter appearing in Eq.(1). We should be careful and not mix $\Delta_0 = 2\Delta$ with it.

where k_x and k_y are the eigenvalues of the momentum operators $\boldsymbol{p}_x$ and $\boldsymbol{p}_y$, respectively. The bare green function that fits the square lattice Hamiltonian is

$$\mathcal{G}_0 = \frac{-\boldsymbol{\tau}_0\omega - \boldsymbol{g_k}\cdot\boldsymbol{\tau}}{-\omega^2 + g_{\boldsymbol{k}}^2 - i\eta} \tag{174}$$

where

$$\boldsymbol{g_k} = \left(2\Delta\sin k_x, 2\Delta\sin k_y, \underbrace{-\frac{1}{m}(\cos k_x + \cos k_y) - \left(\mu - \frac{2}{m}\right)}_{\xi_{\boldsymbol{k}}}\right) \tag{175}$$

and

$$g_{\boldsymbol{k}}^2 = \left[\frac{1}{m}(\cos k_x + \cos k_y) + \left(\mu - \frac{2}{m}\right)\right]^2 + 4\Delta^2\left(\sin^2 k_x + \sin^2 k_y\right) \tag{176}$$

The electron density is

$$n = \lim_{\eta\to 0}\frac{1}{2i}\mathrm{tr}\left(\frac{1}{(2\pi)^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\mathrm{d}^2k\int_{-\infty}^{\infty}\mathrm{d}\omega\; G_0(\boldsymbol{k},\omega)\boldsymbol{\tau}_3 e^{-i\boldsymbol{\tau}_3\eta}\right) = \frac{1}{8\pi^2}\iint_{\square}\mathrm{d}^2k\left(1 - \frac{\xi_{\boldsymbol{k}}}{|\boldsymbol{g_k}|}\right) \tag{177}$$

The coefficient of the partial Chern-Simons term is $-\kappa_a/8\pi$ with[7]:

$$\kappa_a = \frac{1}{4\pi}\int\mathrm{d}^2k\frac{\epsilon_{i\nu\lambda}g_i\partial_{k_x}g_\nu\partial_{k_y}g_\lambda}{|\boldsymbol{g}|^3}. \tag{178}$$

The coefficient of the complete Chern-Simons term is $\kappa_b/8\pi$ with:

$$\kappa_b = \frac{1}{4\pi}\int\mathrm{d}^2k\frac{\epsilon_{\mu\nu\lambda}g_\mu\partial_{k_x}g_\nu\partial_{k_y}g_\lambda}{|\boldsymbol{g}|^3}. \tag{179}$$

All the integrals over the momentum space can be evaluated numerically.

[7] The indices are taken modulo 3, .i.e $3 \longrightarrow 0$

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Appendix B

Signatures of the topological spin of Josephson vortices in topological superconductors

1 The many-body Hamiltonian in momentum space

The Hamiltonian of the topological Josephson junction in the background of a moving soliton is

$$\mathcal{H} = \int \mathrm{d}x \Psi_x^\dagger H \Psi_x, \tag{1}$$

where $\Psi_x = (\psi_x, \bar{\psi}_x)^T$ is a spinor which consists of a periodic and an anti-periodic Majorana field (the fields are real functions), respectively. The single particle Hamiltonian is

$$H = \tau_z i v \partial_x - \tau_y W(x, q), \tag{2}$$

with $W(x) = m\cos[\pi(x-q)/L]$ being the Majorana mass term.

We perform the following mode expansion[1],

$$\psi_x = \frac{1}{\sqrt{L}}\sum_{k_p} e^{-ik_p x}\psi_{k_p} \tag{3}$$

$$\bar{\psi}_x = \frac{1}{\sqrt{L}}\sum_{k_a} e^{ik_a x}\bar{\psi}_{k_a}. \tag{4}$$

where

$$k_p(m) = \frac{2\pi}{L}m, \quad k_a(n) = \frac{2\pi}{L}\left(n+\frac{1}{2}\right), \quad m,n\in\mathcal{Z}. \tag{5}$$

and the opposite signs of the exponents reflect the counter-propagating Majorana edge states. Thus, the Hamiltonian density transforms as follows:

$$\begin{aligned}
\int_{-L/2}^{L/2} \mathrm{d}x\, \psi_x H_{1,1}\psi_x &= \int_{-L/2}^{L/2} \mathrm{d}x\, \psi_x(iv\partial_x)\psi_x = \frac{1}{L}\int_{-L/2}^{L/2}\mathrm{d}x \sum_{k_p,k'_p} e^{-i(k_p+k'_p)x}(vk'_p)\psi_{k_p}\psi_{k'_p} \\
&= \sum_{k_p,k'_p}\delta_{-k_p,k'_p}(vk'_p)\bar{\psi}_{k_p}\psi_{k'_p} = \sum_{k_p}(-vk_p)\psi_{k_p}\psi_{-k_p} = \sum_{n=-N}^{N} vk_p(n)\psi_{-n}\psi_n \\
&= \psi_0\psi_0 + \sum_{n=1}^{N} vk_p\left(\psi_n^\dagger\psi_n - \psi_n\psi_n^\dagger\right)
\end{aligned} \tag{6}$$

[1]If we are interested in the momentum range $k_p(n_{min}) \le k_p \le k_p(n_{max})$ than $k_a(n_{min}) \le k_a \le k_a(n_{max}-1)$.

$$
\begin{aligned}
\int_{-L/2}^{L/2} \mathrm{d}x\, \bar{\psi}_x H_{2,2} \bar{\psi}_x &= \int_{-L/2}^{L/2} \mathrm{d}x\, \bar{\psi}_x (-iv\partial_x) \bar{\psi}_x = \frac{1}{L}\int_{-L/2}^{L/2} \mathrm{d}x \sum_{k_a, k'_a} e^{i(k_a + k'_a)x} (v k'_a) \bar{\psi}_{k_a} \bar{\psi}_{k'_a} \\
&= \sum_{k_a, k'_a} \delta_{-k_a, k'_a} (v k'_a) \bar{\psi}_{k_a} \bar{\psi}_{k'_a} = \sum_{k_a} (-v k_a) \bar{\psi}_{k_a} \bar{\psi}_{-k_a} = \sum_{n=-N}^{N-1} v k_a(n) \bar{\psi}_{-n-1} \bar{\psi}_n \\
&= \sum_{n=0}^{N-1} v k_a(n) \big(\bar{\psi}_n^\dagger \bar{\psi}_n - \bar{\psi}_n \bar{\psi}_n^\dagger \big) \qquad (7) \\
\int_{-L/2}^{L/2} \mathrm{d}x\, \bar{\psi}_x H_{1,2} \psi_x &= \int_{-L/2}^{L/2} \mathrm{d}x\, (\psi_x H_{2,1} \bar{\psi}_x)^\dagger = \int_{-L/2}^{L/2} \mathrm{d}x\, iW(x) \bar{\psi}_x \psi_x \\
&= \frac{im}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x \sum_{k_p, k_a} e^{i(k_a - k_p)x} \cos(\pi \frac{x-q}{L}) \psi_{k_p} \bar{\psi}_{k_a} \\
&= \frac{im}{2L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x \sum_{k_p, k_a} e^{i(k_a - k_p)x} (e^{i\pi \frac{x-q}{L}} + e^{-i\pi \frac{x-q}{L}}) \psi_{k_p} \bar{\psi}_{k_a} \\
&= \frac{im}{2} \sum_{m,n} (\delta_{m,n+1} e^{-i\pi \frac{q}{L}} + \delta_{m,n} e^{i\pi \frac{q}{L}}) \psi_{k_p(m)} \bar{\psi}_{k_a(n)} \\
&= \frac{im}{2} \sum_{m=-N}^{N} \sum_{n=-N}^{N-1} (\delta_{-m,n+1} e^{-i\pi \frac{q}{L}} + \delta_{-m,n} e^{i\pi \frac{q}{L}}) \psi_{-m} \bar{\psi}_n \\
&= \frac{im}{2} \sum_{m=1}^{N} \sum_{n=0}^{N-1} (\delta_{m,n+1} e^{-i\pi \frac{q}{L}} + \delta_{m,n} e^{i\pi \frac{q}{L}}) \psi_m \bar{\psi}_n \\
&\quad + \frac{im}{2} \sum_{m=1}^{N} \sum_{n=0}^{N-1} (\delta_{m,n+1} e^{i\pi \frac{q}{L}} + \delta_{m,n} e^{-i\pi \frac{q}{L}}) \psi_m^\dagger \bar{\psi}_n^\dagger \\
&\quad + \frac{im}{2} (\psi_0 \bar{\psi}_0^\dagger e^{-i\pi \frac{q}{L}} + \psi_0 \bar{\psi}_0 e^{i\pi \frac{q}{L}}) \qquad (8)
\end{aligned}
$$

where we used the relations

$$
k_a(n) = \frac{\pi}{L}(2n+1) = -\frac{\pi}{L}\big(2(-n-1)+1\big) = -k_a(-n-1), \quad k_p(n) = -k_p(-n),
$$

in order to obtain Eq.(6-8).[2]

[2] *Integral representations of Kronecker delta.* For any integer n, using a standard residue calculation we can write an integral representation for the Kronecker delta as the integral below, where the contour of the integral goes counterclockwise around zero. This representation is also equivalent to a definite integral by a rotation in the complex plane. $\delta_{x,n} = \frac{1}{2\pi i} \oint_{|z|=1} z^{x-n-1}\, dz = \frac{1}{2\pi} \int_0^{2\pi} e^{i(x-n)\varphi}\, d\varphi$

Derivation of Eq.(7):

$$
\begin{aligned}
\sum_{k_a}(-vk_a)\bar{\psi}_{k_a}\bar{\psi}_{-k_a} &= \sum_{n=-N}^{N-1}-v\frac{2\pi}{L}\left(n+\frac{1}{2}\right)\bar{\psi}_{k_a(n)}\bar{\psi}_{-k_a(n)} = \\
&\sum_{n=0}^{N-1}-v\frac{2\pi}{L}\left(n+\frac{1}{2}\right)\bar{\psi}_{k_a(n)}\bar{\psi}^{\dagger}_{k_a(n)} + \sum_{n=1}^{N}-v\frac{2\pi}{L}\left(-n+\frac{1}{2}\right)\bar{\psi}_{k_a(-n)}\bar{\psi}_{-k_a(-n)} \\
&\sum_{n=0}^{N-1}-v\frac{2\pi}{L}\left(n+\frac{1}{2}\right)\bar{\psi}_{k_a(n)}\bar{\psi}^{\dagger}_{k_a(n)} + \sum_{n=0}^{N-1}v\frac{2\pi}{L}\left(n+\frac{1}{2}\right)\bar{\psi}_{k_a(-n-1)}\bar{\psi}_{-k_a(-n-1)} = \\
&\sum_{n=0}^{N-1}vk_a(n)\left(\bar{\psi}^{\dagger}_n\bar{\psi}_n-\bar{\psi}_n\bar{\psi}^{\dagger}_n\right) \qquad (9)
\end{aligned}
$$

Derivation of Eq.(8):

$$
\begin{aligned}
&\sum_{\substack{-N<m<0\\-N<n<0}}\left(\delta_{m,n+1}e^{-i\pi\frac{q}{L}}+\delta_{m,n}e^{i\pi\frac{q}{L}}\right)\psi_{k_p(m)}\bar{\psi}_{k_a(n)} = \\
&\sum_{\substack{0<-m<N\\0<-n<N}}\left(\delta_{-m,-n-1}e^{-i\pi\frac{q}{L}}+\delta_{-m,-n}e^{i\pi\frac{q}{L}}\right)\psi_{-k_p(-m)}\bar{\psi}_{-k_a(-n-1)} = \\
&\sum_{\substack{0<m'<N\\0<n'+1<N}}\left(\delta_{m',n'}e^{-i\pi\frac{q}{L}}+\delta_{m',n'+1}e^{i\pi\frac{q}{L}}\right)\psi^{\dagger}_{m'}\bar{\psi}^{\dagger}_{n'} = \sum_{m=1}^{N}\sum_{n=0}^{N-1}\left(\delta_{m,n}e^{-i\pi\frac{q}{L}}+\delta_{m,n+1}e^{i\pi\frac{q}{L}}\right)\psi^{\dagger}_{m}\bar{\psi}^{\dagger}_{n} \qquad (10)
\end{aligned}
$$

The Hamiltonian in terms of Nambu spinors with a cutoff of $N=2\left(|k_p|\le\frac{4\pi}{L}\right)$:

$$
\mathcal{H} = \begin{pmatrix}\psi_{\frac{4\pi}{L}} & \psi_{\frac{2\pi}{L}} & \psi_0 & \psi^{\dagger}_{\frac{2\pi}{L}} & \psi^{\dagger}_{\frac{4\pi}{L}} & \bar{\psi}_{\frac{3\pi}{L}} & \bar{\psi}_{\frac{\pi}{L}} & \bar{\psi}^{\dagger}_{\frac{\pi}{L}} & \bar{\psi}^{\dagger}_{\frac{3\pi}{L}}\end{pmatrix}
\begin{pmatrix}
-\frac{4\pi}{L} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{ic^*}{2} \\
0 & -\frac{2\pi}{L} & 0 & 0 & 0 & 0 & 0 & \frac{ic^*}{2} & \frac{ic}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{ic^*}{2} & \frac{ic}{2} & 0 \\
0 & 0 & 0 & \frac{2\pi}{L} & 0 & \frac{ic^*}{2} & \frac{ic}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{4\pi}{L} & \frac{ic}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{ic}{2} & -\frac{ic^*}{2} & -\frac{3\pi}{L} & 0 & 0 & 0 \\
0 & 0 & -\frac{ic}{2} & -\frac{ic^*}{2} & 0 & 0 & -\frac{\pi}{L} & 0 & 0 \\
0 & -\frac{ic}{2} & -\frac{ic^*}{2} & 0 & 0 & 0 & 0 & \frac{\pi}{L} & 0 \\
-\frac{ic}{2} & -\frac{ic^*}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3\pi}{L}
\end{pmatrix}
\begin{pmatrix}\psi^{\dagger}_{\frac{4\pi}{L}} \\ \psi^{\dagger}_{\frac{2\pi}{L}} \\ \psi_0 \\ \psi_{\frac{2\pi}{L}} \\ \psi_{\frac{4\pi}{L}} \\ \bar{\psi}^{\dagger}_{\frac{3\pi}{L}} \\ \bar{\psi}^{\dagger}_{\frac{\pi}{L}} \\ \bar{\psi}_{\frac{\pi}{L}} \\ \bar{\psi}_{\frac{3\pi}{L}}\end{pmatrix},
$$

where $c\equiv me^{i\pi\frac{q}{L}}$.

Remarks:

1. The spinors in k-space consist of regular fermion fields for $k > 0$ and Majorana fields for $k = 0$.

   Proof:

   We start by calculating the commutation relations,

$$\{\psi_k, \psi_{k'}\} = \sum_{x,x'} e^{-i(kx+k'x)}(\psi_x, \psi_{x'} + \psi_{x'}\psi_x)$$

$$= \sum_x e^{-i(k+k')x} = \delta_{k,-k'} = \begin{cases} 0, & k, k' > 0 \\ 1, & k = k' = 0 \end{cases}. \quad (11)$$

   In addition, since $\psi_x = \sum_k e^{ikx}\psi_k$ and $\psi_x = \psi_x^\dagger$ we find that $\psi_k = \psi_{-k}^\dagger$. Using this relation we find that

$$\{\psi_k, \psi_{k'}^\dagger\} = \{\psi_k, \psi_{-k'}\} = \delta_{k,k'}. \quad (12)$$

   Thus, while for $k = 0$ the fields obey majorana commutation relations, for $k > 0$ they obey fermion commutation relations.

2. The k-space Hamiltonian for reversed boundary conditions can easily deduced by exchanging the positions of ψ_x and $\bar{\psi}_x$ in the spinors and then comparing it with regular Hamiltonian:

$$\begin{aligned} \mathcal{H}_R &= \begin{pmatrix} \psi_x & \bar{\psi}_x \end{pmatrix} (\tau_z iv\partial_x - \tau_y W) \begin{pmatrix} \psi_x \\ \bar{\psi}_x \end{pmatrix} = \begin{pmatrix} \psi_x & \bar{\psi}_x \end{pmatrix} \tau_x^2(\tau_z iv\partial_x - \tau_y W)\tau_x^2 \begin{pmatrix} \psi_x \\ \bar{\psi}_x \end{pmatrix} \\ &= \begin{pmatrix} \bar{\psi}_x & \psi_x \end{pmatrix} (-\tau_z iv\partial_x + \tau_y W) \begin{pmatrix} \bar{\psi}_x \\ \psi_x \end{pmatrix} \end{aligned} \quad (13)$$

   with

$$\bar{\psi}_x = \frac{1}{\sqrt{L}} \sum_{k_p} e^{ik_p x} \bar{\psi}_{k_p} \quad (14)$$

$$\psi_x = \frac{1}{\sqrt{L}} \sum_{k_a} e^{-ik_a x} \psi_{k_a}. \quad (15)$$

   Thus, k-space single particle Hamiltonian for regular boundary condition and the reversed are related by:

$$H_R = \tau_x H \tau_x = -H \quad (16)$$

2 The momentum states of the Hamiltonian density

Next, we derive the connection between the eigenstates in position space and momentum space, of the single particle Hamiltonian. The eigenstates are found by solving the coupled equations,

$$\begin{pmatrix} iv\partial_x & iW(x) \\ -iW(x) & -iv\partial_x \end{pmatrix}\begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = E\begin{pmatrix} f(x) \\ g(x) \end{pmatrix}. \tag{17}$$

We perform the following mode expansion,

$$f(x) = \frac{1}{\sqrt{L}}\sum_{k_p} e^{-ik_p x} f(k_p) \tag{18}$$

$$g(x) = \frac{1}{\sqrt{L}}\sum_{k_a} e^{ik_a x} g(k_a). \tag{19}$$

We substitute the mode expansion and examine a specific k'_p and k'_a by applying the operators $\int_{-L/2}^{L/2} \mathrm{d}x \exp(-ik'_p x)$ and $\int_{-L/2}^{L/2} \mathrm{d}x \exp(ik'_a x)$ to the first and second equations, respectively. This results

$$\begin{aligned} \int \mathrm{d}x \sum_{k_p,k'_p} (-vk_p) f(k_p) e^{-i(k_p+k'_p)x} + i\int \mathrm{d}x \sum_{k_a,k'_p} W(x) g(k_a) e^{i(k_a-k'_p)x} &= E\int \mathrm{d}x \sum_{k_p,k'_p} f(k_p) e^{-i(k_p+k'_p)x} \\ \int \mathrm{d}x \sum_{k_a,k'_a} (-vk_a) g(k_a) e^{i(k_a+k'_a)x} - i\int \mathrm{d}x \sum_{k_p,k'_a} W(x) f(k_p) e^{i(k'_a-k_p)x} &= E\int \mathrm{d}x \sum_{k_a,k'_a} g(k_a) e^{i(k_a+k'_a)x} \end{aligned}$$

Preforming the integration with respect to x yields

$$-v\sum_{k_p} k_p f(k_p) + i\sum_{k_p,k_a} W(k_a-k_p) g(k_a) = E\sum_{k_p} f(k_p) \tag{20}$$

$$-v\sum_{k_a} k_a g(k_a) - i\sum_{k_a,k_p} W(k_a-k_p) f(k_p) = E\sum_{k_a} g(k_a) \tag{21}$$

where for brevity we omitted the prime tag and $W(k) \equiv \frac{1}{L}\int_{-L/2}^{L/2} \mathrm{d}x\, e^{ikx} W(x)$.

3 Properties of the Hamiltonian in k-space

The Hamiltonian in k-space posses the following properties:

1. Particle-Hole symmetry (PHS),
$$H = -(C_{n+1} \oplus C_n)\mathbf{K}H\mathbf{K}(C_{n+1} \oplus C_n), \tag{22}$$
where $\mathbf{K}$ is the complex conjugation operator and C_n is anti-diagonal matrix of ones with dimension n[3].

2. Reversed soliton symmetry (RSS),
$$H(-q) = (\tau_z\mathbf{K})H(q)(\mathbf{K}\tau_z), \tag{23}$$
which is valid since $c(-q) = -c^*(q)$ with $c \equiv me^{i\pi\frac{q}{L}}$.[4]

3. For any set of parameters (v, m, L, q), there will be a single zero-mode,
$$\det(H) = 0. \tag{24}$$
Moreover, the zero-mode is an eigenstate of the PHS operator.

4. For $m = const$ the eigenstates are periodic in q with a period of $2L$. This can easily be observed by writing the Hamiltonian as,
$$H(q) = T(q)H(q = 0)T^{\dagger}(q), \tag{25}$$
where $T = T^{(1)} \oplus T^{(2)}$ is a diagonal block matrix with
$$T^{(1)}_{m,n} = \delta_{m,n}e^{i\pi\frac{q}{L}(2m-1)}e^{-i\pi\frac{q}{L}(2N)}, \qquad 1 \le m, n \le N + 1 \tag{26}$$
$$T^{(2)}_{m,n} = \delta_{m,n}e^{-i\pi\frac{q}{L}(2m-1)}e^{i\pi\frac{q}{L}(2N+1)}, \quad 1 \le m, n \le N. \tag{27}$$
Thus, the eigenstates of the system are $T^{\dagger}(q)\psi(q = 0)$.

5. For $m = const$ the spectrum of the system does not depend on the soliton's displacement, q. We notice that T is also unitary which means that $H(q)$ can be expressed as similarity transformation of $H(q = 0)$ and the eigenvalues of matrix are always invariant under similarity transformations.

[3]An n-by-n matrix A is an anti-diagonal matrix if the (i, j) element is zero for all $i, j \in 1, \ldots, n$ with $i + j \neq n + 1$.
[4]$c = const$ fulfills this condition.

4 Choose a Majorana mass term that retain the periodicity of the soliton

As explained in section 3, the soliton returns to its original form after it completes two cycles while it is expected to occur each cycle. In terms of the system eigenstate $\psi(q) = (u, v)^T$ we have

$$\begin{pmatrix} u(q+L) \\ v(q+L) \end{pmatrix} = \begin{pmatrix} u(q) \\ -v(q) \end{pmatrix}, \tag{28}$$

where u and v are vectors that correspond to the PHS relation which appears in Eq.(22). The soliton would return to its original form after one cycle if the eigenstate would be

$$\psi(q) = \begin{pmatrix} U(q) \\ (-1)^{\lfloor \frac{q}{L} + \frac{1}{4} \rfloor} V(q) \end{pmatrix}. \tag{29}$$

This can be adding to the Majorana mass term a branch-cut,

$$W(x) \longrightarrow (-1)^{\lfloor \frac{q}{L} + \frac{1}{4} \rfloor} W(x), \tag{30}$$

where the shift of floor's argument is due to our demand that two counter propagating solitons would consolidate twice in a cycle and their overlap would change smoothly. In a addition the new order-parameter assures us that the spinors are single-valued with respect to the soliton displacement.

The connection between the desired eigenstate in Eq.(29) and the suggested Majorana mass term in Eq.(30) is made by expressing the corresponding Hamiltonian as a similarity transformation,

$$H(q) = (Z(q)T(q))\, H(q=0) \left(T^\dagger(q) Z^\dagger(q)\right), \tag{31}$$

where T was defined in Eq.(26) and $Z = Z^{(1)} \oplus Z^{(2)}$ is a diagonal block matrix with

$$\begin{aligned} Z^{(1)}_{m,n} &= \delta_{m,n}, & 1 \le m, n \le 2n+1 \\ Z^{(2)}_{m,n} &= \delta_{m,n}(-1)^{\lfloor \frac{q}{L} + \frac{1}{4} \rfloor}, & 1 \le m, n \le 2n. \end{aligned} \tag{32}$$

Thus, the system's eigenstates are $\psi(q) = Z(q)T(q)\psi(q=0)$.

One may raise the question - why not simply to make the the Majorana mass term to generate a $U(1)$ group, such as $W(x) \longrightarrow e^{\pm i\pi \frac{q}{L}} W(x)$. Actually we can consider a more general Majorana mass term, $W(x) \longrightarrow e^{\pm i(2j+1)\pi \frac{q}{L}} W(x)$ with $j \in \mathbb{Z}$. The corresponding similarity transformation is

$$H(q) = (M(q)T(q))\, H(q=0) \left(T^\dagger(q) M^\dagger(q)\right), \tag{33}$$

where $M = M^{(1)} \oplus M^{(2)}$ is a diagonal block matrix,

$$\begin{aligned} M^{(1)}_{m,n} &= \delta_{m,n}, & 1 \le m, n \le 2n+1 \\ M^{(2)}_{m,n} &= \delta_{m,n} e^{\mp i(2j+1)\pi \frac{q}{L}}, & 1 \le m, n \le 2n. \end{aligned} \tag{34}$$

However, this transformation results unphysical eigenstates, $\psi(q) = M(q)T(q)\psi(q=0)$ as it renders eigenstates of well separated solitons to overlap.

5 Add the missing Majorana fermion

The spinor contains a single zero-momentum field and $2n$ pairs of counter-propagating fields with non-zero momentum. The zero-momentum field is a Majorana fermion that is located on one of the junction edges. Its position will depend on the parity of the number of vortices enclosed by the junction. A zero-momentum Majorana is found on every edge that encloses an odd number of vortices (including both bulk vortices and solitons). Hence, when we have an odd parity, the zero-momentum Majorana will be localized at the inner edge of the junction. On the other hand, for an even parity the zero-momentum Majorana will be localized on the outer edge of the junction. A second zero-momentum field, which is absent in our one dimensional effective model, is localized around the core of one of the bulk vortices or at the edge that encloses the whole physical system, depending on the parity. Since our model is an effective theory of the Josephson junction, the second zero-momentum Majorana is absent. Thus, in order to recover the PHS we need to add, by hand, an uncoupled zero momentum field to our model.

In order to add a decoupled zero-momentum Majorana to our model we add a decoupled block to the Hamiltonian, which is just zero. In practice, we use the transformation $\Psi \to P\Psi$
with

$$P = \begin{pmatrix} I_N & 0 \\ 0 & 0 \\ 0 & I_{3N+1} \end{pmatrix} \tag{35}$$

and I_n is an identity matrix of dimension n.

6 Transform the two Majorana fermions into a regular fermions

As shown in section 1, fields of zero momentum are majorana fermions. In order to write the groundstate in Thouless represetaion, we transform the two majorana fields into regular fermion fields, $\Psi \to TC\Psi$ with

$$T = I_n \oplus \frac{1}{\sqrt{2}} \begin{pmatrix} I & 1 \\ -I & 1 \end{pmatrix} \oplus I_{3n}. \tag{36}$$

7 Transform the Hamiltonian into a "standard" PHS form

We would like to transform the single particle Hamiltonian such that PHS would take the form,

$$-H = (\tau_x \mathbf{K}) H (\mathbf{K} \tau_x). \tag{37}$$

where $\mathbf{K}$ is the complex conjugation operator. We limit ourselves to transformations that merely change the fields order in the spinor, since we do not want to mix between fields that correspond to different momentums. These requirements are accomplished by the transformation $\Psi \to WTC\Psi$ with

$$W_{m,n} = \begin{cases} \delta_{\lceil \frac{2N+2+m}{2} \rceil + (2N+1)\mathrm{mod}(2N+1+m,2), n}, & m \leq 2N+1 \\ \delta_{\lceil \frac{4N+3-m}{2} \rceil + (2N+2)\mathrm{mod}(4N+4-m,2), j}, & 2N+1 < m \end{cases}. \tag{38}$$

8 Representing our new Hamiltonian with a similarity transformation that depend on q

The single particle Hamiltonian that correspond to a spinor of fermions with a PHS, $H'(q) = (WTC)H(q)(C^\dagger T^\dagger W^\dagger)$, can be expressed as a similarity transformation of $H'(0)$,

$$H'(q) = (Z(q)P(q))\, H'(0) \left(P^\dagger(q) Z^\dagger(q)\right). \tag{39}$$

The matrix $P = P^{(1)} \oplus P^{(2)}$ is unitary with

$$P^{(1)}_{m,n} = e^{(-1)^m (1-m) \frac{i\pi q}{L}} \delta_{m,n}, \quad m \leq 2N+1 \tag{40}$$

$$P^{(2)}_{m,n} = e^{(-1)^m (m-1) \frac{i\pi q}{L}} \delta_{m,n}, \quad m \leq 2N+1 \tag{41}$$

and $Z = Z^{(1)} \oplus Z^{(2)}$ is orthogonal with

$$Z^{(1)}_{m,n} = Z^{(2)}_{m,n} = (-1)^{\mathrm{mod}(m,2) \lfloor \frac{q}{L} + \frac{1}{4} \rfloor} \delta_{m,n}, \quad m \leq 2N+1. \tag{42}$$

9 Reversing the boundary conditions

As described in the remarks of section 1, by reversing the boundary conditions of ψ and $\bar{\psi}$ the single particle Hamiltonian transforms as $H \to -H$. Consequently, the transformations, P and T that adds a second majorana to the model and then transform the two majoranas into regular fermions are kept unchanged. However, the transformation, $W^{(R)}$ that brings the Hamiltonian into a "standard" PHS form is obtained from the former transformation, W by a circular shift of $2N+1$ rows:

$$W^{(R)}_{i,j} = \begin{cases} \delta_{\lceil \frac{2n+2-i}{2} \rceil + (2n+2)\text{mod}(2n+3-i,2),j} & i \leq 2n+1 \\ \delta_{\lceil \frac{i+1}{2} \rceil + (2n+1)\text{mod}(i,2),j} & 2n+1 < i \end{cases}. \tag{43}$$

Hence, the single particle Hamiltonian that correspond to a spinor of fermions with reversed boundary conditions and PHS is

$$H^{(R)}(q) = -(W^{(R)}TC)H(q)(C^\dagger T^\dagger W^{(R)\dagger}). \tag{44}$$

In addition, the reversed Hamiltonian, $H^{(R)}(q)$ can be expressed as a similarity transformation of $H^{(R)}(0)$,

$$H^{(R)}(q) = \left(Z(q)P^{(R)}(q)\right) H^{(R)}(0) \left(P^{(R)\dagger}(q)Z^\dagger(q)\right). \tag{45}$$

where $P^{(R)}$ is obtained from P by interchanging its two blocks,

$$P^{(R)} = (\tau_x \otimes I_{2N+1})\, P\, (\tau_x \otimes I_{2N+1}) = P^{(2)} \oplus P^{(1)} \tag{46}$$

and $Z = Z^{(1)} \oplus Z^{(1)}$ is evidently unchanged.

10 The Geometric phase that groundstate acquire during the soliton's motion

The Berry connection for a BCS many-body states is given by Read's formula[2]:

$$i\langle \Omega_q | \partial_q \Omega_q \rangle = \frac{i}{4} \text{tr}\left((1 + Z^\dagger Z)^{-1}(Z^\dagger Z' - Z'^\dagger Z)\right) \tag{47}$$

where $|\Omega\rangle$ is the many-body groundstate ,$Z = (VU^{-1})^*$ and the columns of the block matrix $(U\ V)^T$ are eigenstates that correspond to positive eigenenergies in an acceding order. Read's formula can be brought into a more suitable form for taking symbolically the derivatives,

$$i\langle \Omega_q | \partial_q \Omega_q \rangle = \frac{i}{4} \text{tr}\left[V^\dagger V' - V\left(V^\dagger\right)' + \left(U^\dagger\right)^{-1} V^\dagger V \left(U^\dagger\right)' - V^\dagger V U^{-1} U'\right]^*, \tag{48}$$

in which only derivatives of the matrices U and V appear.

The standard procedure for calculating the overlap between two many-body states assume that all positive energy single-particle eigenstates, that form $(U\ V)^T$, are related to the negative ones by the PHS operator, $\tau_x K$. However, it is not granted that degenerate eigenstates would obey this relation and one must construct such states (as described in the frame below, titled "Generating PHS zero-modes"). In order to construct the many-body groundstate, we choose the zero-mode which leads to a non-vanishing determinant of U to correspond to a positive energy state (as explained in

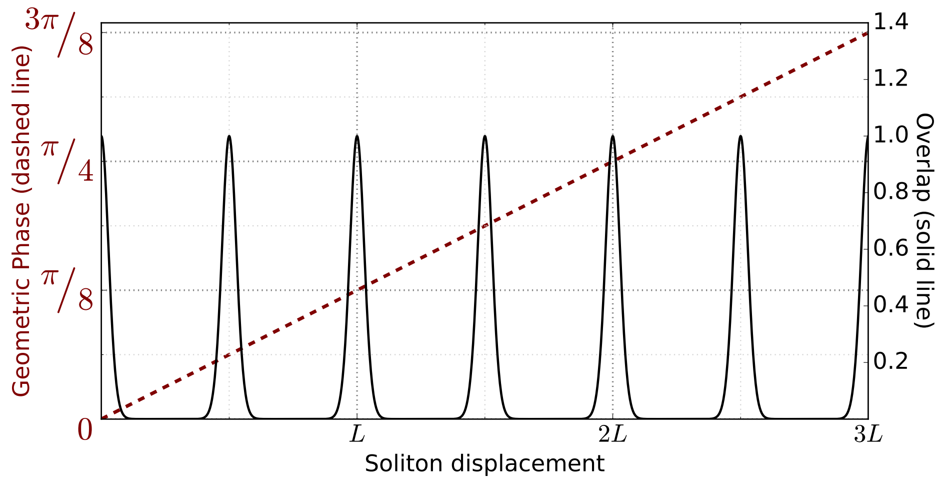

Figure 1: **The geometric phase accumulated by the persisting Josephson vortex**. The dashed (brown) line describes the geometric phase accumulated by each persisting soliton in the presence of a vortex within the central region. In addition, the solid (black) line describes the overlap norm of two counter-propagating solitons, which becomes non zero at half cycles. At these points the geometric phase of each soliton acquires its universal values $n\pi/16$, $n \in \mathbb{Z}$.

the frame below, titled "Building the groundstate in the presence of zero-mode").

Using the procedure discussed above, we diagonalize H numerically for $q = 0$ and using the translation operator $Z(q)P(q)$ we obtain the eigenvectors for any other position of the soliton. We substitute into Eq. 48, performing the derivative symbolically. The result is presented in Fig. 1 with the overlap calculated using the Onishi formula, $|\langle\Omega_{-q}|\Omega_q\rangle| = \sqrt{|\det \chi^\dagger_{-q}\chi_q|}$ with $\chi_q = T_q\chi_0$ and $\chi_0^T = (U_0^T, V_0^T)$ [1] for two counter-persisting solitons, demonstrating that the topological spin is in principle an observable. We repeated the procedure taking reversed boundary conditions on the two Majorana edge states, obtaining the same phase but with an additional minus sign,which up to machine precision is $-\pi/8L$.

Generating PHS zero-modes

Here describe a procedure to construct zero-modes which are related by the PHS operator. The PHS, $H = \tau_x H^* \tau_x$, assures us that two non-degenerate states with energy $|\epsilon| > 0$ are related by $|\epsilon\rangle = \tau_x K|-\epsilon\rangle$. In the case of zero-modes, $\epsilon = 0$ we need to construct two orthonormal states that also maintain this relation,

$$|v_1\rangle = a|u_1\rangle + b|u_2\rangle \quad (49)$$

$$|v_2\rangle = Tv_1 = \bar{a}T|u_1\rangle + \bar{b}T|u_2\rangle \quad (50)$$

where u_1 and u_2 are two arbitrary orthonormal zero-modes that fulfill $\langle u_1|T|u_2\rangle = 0$, T is the PHS operator, $T = \tau_x K$ (K is complex conjugate operator). Moreover, u_1 and u_2 are simply

$$u_1 = WTCu'_1, \quad (51)$$

$$u_2 = WTCu'_2, \quad (52)$$

where u'_1 is the single zero-mode of $H(q)$, and $(u'_2)_i = \delta_{i,N+1}$ is the uncoupled zero-mode that was added to the model. The requirement that v_1 (and v_2) are normalized gives a constrain on a and b,

$$\langle v_1|v_1\rangle = 1 \Rightarrow |a|^2 + |b|^2 = 1 \Rightarrow a = |\cos\alpha|e^{i\beta}, \quad b = |\sin\alpha|e^{i\gamma}. \quad (53)$$

The second requirement, namely, that the two zero-mode are orthogonal yields the following constrain:

$$\langle v_1|v_2\rangle = \bar{a}^2\langle u_1|T|u_1\rangle + \bar{b}^2\langle u_2|T|u_2\rangle + \bar{a}\bar{b}\left(\langle u_1|T|u_2\rangle + \langle u_2|T|u_1\rangle\right) = 0. \quad (54)$$

Noticing that

$$\langle u_2|T|u_1\rangle = \langle u_1|T|u_2\rangle \quad (55)$$

the relation can be further simplified,

$$\langle v_1|v_2\rangle = \bar{a}^2A + \bar{b}^2B + \bar{a}\bar{b}C = 0. \quad (56)$$

where $A = \langle u_1|T|u_1\rangle$, $B = \langle u_2|T|u_2\rangle$ and $C = 2\langle u_1|T|u_2\rangle$. Since $C = 0$, the constrain is simplified further to $A\bar{a}^2 + B\bar{b}^2 = 0$ and together with the first constrain, $|a|^2 + |b|^2 = 1$ we get

$$|a|^2 = \frac{1}{1 - \frac{A}{B}e^{i2(\gamma-\beta)}} \quad (57)$$

with $\gamma - \beta = -\frac{1}{2}\arg\frac{A}{B} + (n + \frac{1}{2})\pi$ and $n \in \mathbb{Z}$ because $|a| < 1$. Since the zero-modes are defined up to a phase it's enough to determine the relative phase between a and b. Out of the infinite possibilities to choose the phases β and γ, it is convenient to pick $\beta = \frac{1}{2}\arg A$ and $\gamma = \frac{1}{2}\arg B$.

Building the groundstate in the presence of zero-mode

In order to construct the many-body groundstate, we start with the bare vacuum $|\Omega\rangle$ and multiply it by a product of quasi-particle annihilation operators, c_ϵ with positive energy. In the end we multiply it with one of the zero-mode operators,

$$|\Omega\rangle = c_{0\pm} \prod_j c_{\epsilon_i} |0\rangle. \tag{58}$$

One of the two zero-mode operators would make the construed state vanish identically. This point is better understood by considering the Thouless representation of the groundstate,

$$|\Omega\rangle = \sqrt{|\det U|} \exp\left(\sum_{i<j} Z_{ij} \psi_i^\dagger \psi_j^\dagger \right) |0\rangle, \quad Z = (VU^{-1})^*. \tag{59}$$

When the groundstate vanishes identically, $\det U = 0$ and since $\langle 0|\Omega\rangle = \sqrt{|\det U|}$ it means that groundstate is orthogonal to the bare vacuum. In addition, $\det U = 0$ means that U is singular and Z is undefined. Practically, we identify the zero-mode for which $\det U \neq 0$ as the annihilation operator and use it to contract the groundstate.

References

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Appendix C

How vortex bound states affect the Hall conductivity of a chiral $p \pm ip$ superconductor

1 The cylindrical argument function

In this section we adjust the argument function to a flat right-angled cylinder of circumference $q_y > 0$, denoted by C. In order to do so, we calculate the principal value of

$$\lim_{A\to\infty} \sum_{n=-2A}^{2A} \mathrm{Arg}\left(z + inq_y\right), \tag{1}$$

which turns out to be convergent in $\mathbb{R}/2\pi\mathbb{Z}$. Furthermore, the infinite series converges into an elegant expression. Thus, we define it as the cylindrical argument function:

$$\phi^C(z, iq_y) \equiv \mathcal{P}.\mathcal{V}.\sum_{n=-\infty}^{\infty} \mathrm{Arg}\left(z + inq_y\right) = \mathrm{Im}\left[\mathrm{Log}\sinh\left(\frac{\pi z}{q_y}\right)\right]. \tag{2}$$

Derivation of the cylindrical argument function. We consider the following series in $\mathbb{R}/\pi\mathbb{Z}$,

$$\sum_{n=-A}^{A} \text{Arg}\left(x+i\left(y+nq_y\right)\right) = \frac{i}{2}\sum_{n=-A}^{A}\left[\text{Log}\left(x-i\left(y+nq_y\right)\right)-\text{Log}\left(x+i\left(y+nq_y\right)\right)\right]$$
$$\equiv \frac{i}{2}\sum_{n=-A}^{A}\left[\text{Log}\left(i\bar{z}+n\right)-\text{Log}\left(iz+n\right)\right] \pmod{\pi}, \tag{3}$$

where $z=\frac{x+iy}{q_y}$.

Recall that for any $z\in\mathbb{C}$

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}, \tag{4}$$

it follows that

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2\prod_{n\neq 0}\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}.[1] \tag{5}$$

On the other hand, Weierstrass factorization theorem states that

$$\sin(\pi z) = \pi z\prod_{n\neq 0}\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}. \tag{6}$$

Combining the above yields

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -\frac{z\sin(\pi z)}{\pi}. \tag{7}$$

Plugging this into the identity in Eq.(95) and applying it to Eq.(3) yields

$$\sum_{n=-A}^{A}\text{Arg}\left(x+i\left(y+nq_y\right)\right)$$
$$\equiv \frac{i}{2}\left[\text{Log}\left(\frac{\bar{z}\sinh(\pi\bar{z})}{z\sinh(\pi z)}\right)+\text{Log}\left(\frac{\Gamma(1+A+i\bar{z})\Gamma(1+A-i\bar{z})}{\Gamma(1+A+iz)\Gamma(1+A-iz)}\right)-\text{Log}\left(\frac{\bar{z}}{z}\right)\right] \pmod{\pi}$$
$$\equiv \frac{i}{2}\left[\text{Log}\left(\frac{\sinh(\pi\bar{z})}{\sinh(\pi z)}\right)+\text{Log}\left(\frac{\Gamma(1+A+i\bar{z})\Gamma(1+A-i\bar{z})}{\Gamma(1+A+iz)\Gamma(1+A-iz)}\right)\right] \pmod{\pi}. \tag{8}$$

We now note that

$$\lim_{A\to\infty}\frac{\Gamma(1+A+i\bar{z})\Gamma(1+A-i\bar{z})}{\Gamma(1+A+iz)\Gamma(1+A-iz)} = 1.$$

Thus,

$$\lim_{A\to\infty}\sum_{n=-A}^{A}\text{Arg}\left(z+inq_y\right) \equiv \frac{i}{2}\text{Log}\left(\frac{\sinh\left(\frac{\pi\bar{z}}{q_y}\right)}{\sinh\left(\frac{\pi z}{q_y}\right)}\right) \pmod{\pi} \tag{9}$$
$$\equiv \frac{i}{2}\left(\text{Log}\sinh\left(\frac{\pi\bar{z}}{q_y}\right)-\text{Log}\sinh\left(\frac{\pi z}{q_y}\right)\right) \pmod{\pi}, \tag{10}$$

where here $z\equiv x+iy$. In addition, in our last step we performed a lift so the infinite sum would converge into a function that is well defined in $\mathbb{R}/2\pi\mathbb{Z}$, just like the argument functions.

While $\phi^C\left(z, q_y\right)$ itself is not well defined on the cylinder C, it serves as a building block for multi-singularity configurations,

$$\Theta^C(z) \equiv \sum_i n_i \phi^C\left(z - z_i, iq_y\right), \tag{11}$$

that are iq_y-periodic modulo 2π. Here $z_i \in \mathbb{C}$ and $n_i \in \mathbb{Z}$ are the displacement and class of the i^{th} singularity, respectively. In addition, we have the selection rule,

$$\sum_i n_i = 0, \quad (\text{mod } 2) \tag{12}$$

that determines which multi-singularity configurations are supported by the cylindrical topology.

Derivation of the selection rule. We aim to check how the cylindrical argument function transforms under $x \longrightarrow x + q_x$:

$$\text{Log}\sinh\left(\frac{\pi}{q_y}(x + iy + iq_y)\right) = \text{Log}\left(e^{\frac{\pi}{q_y}(x+iy+iq_y)} - e^{-\frac{\pi}{q_y}(x+iy+iq_y)}\right) - \text{Log}\,2 =$$
$$\text{Log}\left[e^{i\pi}\left(e^{\frac{\pi}{q_y}(x+iy)} - e^{-\frac{\pi}{q_y}(x+iy+2iq_y)}\right)\right] - \text{Log}\,2 = \text{Log}\left[e^{i\pi}\left(e^{\frac{\pi}{q_y}(x+iy)} - e^{-\frac{\pi}{q_y}(x+iy)}\right)\right] - \text{Log}\,2 + i\pi(2n+1)$$

Thus, we find that $\phi^C(z + iq_y, iq_y) = \phi^C(z, iq_y) + \pi$. This is in contrary to our exception that the phase function $\phi^C\left(z, q_y\right)$ would be iq_y-periodic.

An example of a configuration that results a iq_y-periodic phase field is the placement of a single vortex anti-vortex (V-AV) pair on a cylinder,

$$\Theta(z) = \phi^C\left(z - z_1, iq_y\right) - \phi^C\left(z - z_2, iq_y\right), \tag{13}$$

with $z, z_1, z_2 \in \mathbb{C}$.

The cylindrical argument function can be generalized to support periodicity along any axis, $\mathbb{C}/\mathbb{Z}\tau$ by substituting $q_y = -i\tau$ in Eq.(2):

$$\phi^C(z, \tau) \equiv \mathcal{P}.\mathcal{V}.\sum_{n=-\infty}^{\infty}\text{Arg}(z+n\tau) = \text{Im}\left[\text{Log}\sinh\left(\frac{i\pi z}{\tau}\right)\right], \tag{14}$$

where $z, \tau \in \mathbb{C}$ and $z \equiv x+iy$. The norm $|\tau|$ stands for the cylinder circumference while the phase $\text{Arg}(\tau)$ represent the angle between the periodic boundary condition axis and the x-axis. In addition, the cylindrical argument function obeys the relation $\phi^C(z + \tau, \tau) = \phi^C(z, \tau) + \pi$ and thus the selection rule in Eq.(12) holds also for acute-angled flat cylinders.

The name "cylindrical argument function" is justified by the fact that on one hand it is a singly periodic function and on the other hand it approaches the argument function in a vicinity of a vortex, $\phi^C\left(re^{i\phi}, \tau\right) \xrightarrow[r\to 0]{} \phi$.

2 The toroidal argument function

In this section we adjust the argument function to a flat right-angled torus of circumferences $q_x, q_y > 0$, denoted by T. In order to do so, we calculate the principal value of

$$\lim_{A\to\infty}\sum_{m,n=-2A}^{2A}\text{Arg}\left(z + mq_x + inq_y\right)$$

which turns out to be convergent in $\mathbb{R}/2\pi\mathbb{Z}$. Furthermore, the infinite series converges into an elegant expression. Thus, we define it as the toroidal argument function:

$$\phi^T(z, q_x, -iq_y) \equiv \mathcal{P}.\mathcal{V}. \lim_{A\to\infty} \sum_{m,n=-2A}^{2A} \mathrm{Arg}\left(z+mq_x+inq_y\right) = \mathrm{Im}\left[\mathrm{Log}\left(i\vartheta_1\left(\frac{z}{iq_y}, \frac{iq_x}{q_y}\right)\right) - \frac{2z^2}{q_xq_y}\mathrm{arctg}\left(\frac{q_x}{q_y}\right)\right], \tag{15}$$

where $q_x, q_y > 0$ and arctg is the arctangent function (with an image in $[-\pi/2, \pi/2]$).

Derivation of the toroidal argument function. We consider the following series in $\mathbb{R}/\pi\mathbb{Z}$,

$$\sum_{m,n=-A}^{A} \mathrm{Arg}\left(x+mq_x+i\left(y+nq_y\right)\right) = \sum_{m=-A}^{A}\left(\sum_{n=-A}^{A} \mathrm{Arg}\left(x+mq_x+i\left(y+nq_y\right)\right)\right), \tag{16}$$

where we choose to first sum over n and only then over m. Applying Eq.(8) yields

$$\sum_{m=-A}^{A}\left(\sum_{n=-A}^{A} \mathrm{Arg}\left(x+mq_x+i\left(y+nq_y\right)\right)\right) \equiv$$
$$\sum_{m=-A}^{A} \frac{i}{2}\left[\mathrm{Log}\left(\frac{\sinh(\pi\bar{z}_m)}{\sinh(\pi z_m)}\right) + \mathrm{Log}\left(\frac{\Gamma(1+A+i\bar{z}_m)\,\Gamma(1+A-i\bar{z}_m)}{\Gamma(1+A+iz_m)\,\Gamma(1+A-iz_m)}\right)\right] \pmod{\pi}, \tag{17}$$

where $z_m = \frac{x+mq_x+iy}{q_y}$. We start by considering the following sum

$$\sum_{m=-A}^{A}\mathrm{Log}\left(\frac{\sinh(\pi\bar{z}_m)}{\sinh(\pi z_m)}\right) \equiv \mathrm{Log}\left(\prod_{m=-A}^{A}\frac{\sinh(\pi\bar{z}_m)}{\sinh(\pi z_m)}\right) \pmod{\pi}.$$

It holds that

$$\prod_{m=-A}^{A}\frac{\sinh(\pi\bar{z}_m)}{\sinh(\pi z_m)} = \prod_{m=-A}^{A}\frac{e^{\pi\bar{z}_m}-e^{-\pi\bar{z}_m}}{e^{\pi z_m}-e^{-\pi z_m}} = \frac{e^{\pi z_0}-e^{\pi z_0}}{e^{\pi\overline{z_0}}-e^{\pi\overline{z_0}}}\prod_{m=0}^{A}\left(\frac{e^{\pi\bar{z}_m}\left(1-e^{-2\pi\bar{z}_m}\right)}{e^{\pi z_m}\left(1-e^{-2\pi z_m}\right)}\right)\left(\frac{e^{-\pi\bar{z}_{-m}}\left(1-e^{2\pi\bar{z}_{-m}}\right)}{e^{-\pi z_{-m}}\left(1-e^{2\pi z_{-m}}\right)}\right)$$
$$= \frac{\sinh(\pi z_0)}{\sinh(\pi\overline{z_0})}\prod_{m=0}^{A}\frac{\left(1-e^{-2\pi\bar{z}_m}\right)\left(1-e^{2\pi\bar{z}_{-m}}\right)}{\left(1-e^{-2\pi z_m}\right)\left(1-e^{2\pi z_{-m}}\right)}. \tag{18}$$

Using Eq.(83) one checks that

$$\prod_{m=-\infty}^{\infty}\frac{\sinh(\pi\bar{z}_m)}{\sinh(\pi z_m)} = e^{\frac{2\pi iy}{q_y}}\frac{\vartheta_3\left(\frac{ix+y}{q_y}+\frac{iq_x+q_y}{2q_y}, \frac{iq_x}{q_y}\right)}{\vartheta_3\left(\frac{ix-y}{q_y}+\frac{iq_x+q_y}{2q_y}, \frac{iq_x}{q_y}\right)}.$$

It follows that

$$\frac{i}{2}\sum_{m=-\infty}^{\infty}\mathrm{Log}\left(\frac{\sinh(\pi\bar{z}_m)}{\sinh(\pi z_m)}\right) \equiv \frac{i}{2}\mathrm{Log}\left(e^{\frac{2\pi iy}{q_y}}\frac{\vartheta_3\left(\frac{ix+y}{q_y}+\frac{iq_x+q_y}{2q_y}, \frac{iq_x}{q_y}\right)}{\vartheta_3\left(\frac{ix-y}{q_y}+\frac{iq_x+q_y}{2q_y}, \frac{iq_x}{q_y}\right)}\right) \pmod{\pi}.$$

We now turn to deal with

$$\lim_{A\to\infty}\sum_{m=-A}^{A}\operatorname{Log}\left(\frac{\Gamma(1+A+i\bar{z}_m)\,\Gamma(1+A-i\bar{z}_m)}{\Gamma(1+A+iz_m)\,\Gamma(1+A-iz_m)}\right). \tag{19}$$

In what follows, we assume that for any $A\in\mathbb{N}$

$$\lim_{A\to\infty}\sum_{m=-A}^{A}\operatorname{Log}\left(\frac{\Gamma(1+A+i\bar{z}_m)\,\Gamma(1+A-i\bar{z}_m)}{\Gamma(1+A+iz_m)\,\Gamma(1+A-iz_m)}\right)=\lim_{A\to\infty}\int_{-A}^{A}\operatorname{Log}\left(\frac{\Gamma(1+A+i\bar{z}_m)\,\Gamma(1+A-i\bar{z}_m)}{\Gamma(1+A+iz_m)\,\Gamma(1+A-iz_m)}\right)\mathrm{d}m.$$

Recall from [2, Eq.11] that the for any $z\in\mathbb{C}$ with $\operatorname{Re}(z)>0$ it holds that

$$\int_0^z\operatorname{Log}\Gamma(x)\,\mathrm{d}x=\frac{(1-z)\,z}{2}+\frac{z}{2}\log(2\pi)-\zeta'(-1)+\zeta_z'(-1), \tag{20}$$

where $\zeta(s)$ and $\zeta_a(s)$ are Riemann zeta and Hurwitz zeta functions ,respectively. We write

$$J_1(A)=\int_{-A}^{A}\operatorname{Log}\Gamma\left(A+1-\frac{i\,(x+mq_x)+y}{q_y}\right)\mathrm{d}m, \tag{21}$$

$$J_2(A)=\int_{-A}^{A}\operatorname{Log}\Gamma\left(A+1+\frac{i\,(x+mq_x)+y}{q_y}\right)\mathrm{d}m, \tag{22}$$

$$J_3(A)=\int_{-A}^{A}\operatorname{Log}\Gamma\left(A+1+\frac{i\,(x+mq_x)-y}{q_y}\right)\mathrm{d}m, \tag{23}$$

$$J_4(A)=\int_{-A}^{A}\operatorname{Log}\Gamma\left(A+1-\frac{i\,(x+mq_x)-y}{q_y}\right)\mathrm{d}m. \tag{24}$$

Inserting these and Eq.(20) into Eq.(19) and taking the limits $A\longrightarrow\infty$ yields

$$\frac{i}{2}\sum_{m=-\infty}^{\infty}\operatorname{Log}\left(\frac{\Gamma(1+A+i\bar{z}_m)\,\Gamma(1+A-i\bar{z}_m)}{\Gamma(1+A+iz_m)\,\Gamma(1+A-iz_m)}\right)\equiv-\frac{4xy}{q_xq_y}\operatorname{arctg}\left(\frac{q_x}{q_y}\right)\pmod{\pi}. \tag{25}$$

Combining the parts, reveals that the infinite sums converge into an elegant expression:

$$\begin{aligned}
&\lim_{A\to\infty}\sum_{m=-A}^{A}\sum_{n=-A}^{A}\operatorname{Arg}\left(x+mq_x+i\left(y+nq_y\right)\right)\equiv \qquad (26)\\
&\frac{i}{2}\operatorname{Log}\left(\frac{\vartheta_1\left(\frac{ix+y}{q_y},\frac{iq_x}{q_y}\right)}{\vartheta_1\left(\frac{ix-y}{q_y},\frac{iq_x}{q_y}\right)}\right)-\frac{4xy}{q_xq_y}\operatorname{arctg}\left(\frac{q_x}{q_y}\right)\pmod{\pi}\equiv\\
&\frac{i}{2}\left[\operatorname{Log}\left(i\vartheta_1\left(\frac{-ix-y}{q_y},\frac{iq_x}{q_y}\right)\right)-\operatorname{Log}\left(i\vartheta_1\left(\frac{-ix+y}{q_y},\frac{iq_x}{q_y}\right)\right)\right]-\frac{4xy}{q_xq_y}\operatorname{arctg}\left(\frac{q_x}{q_y}\right)\pmod{\pi}=\\
&\operatorname{Im}\left[\operatorname{Log}\left(i\vartheta_1\left(\frac{-ix+y}{q_y},\frac{iq_x}{q_y}\right)\right)\right]-\frac{4xy}{q_xq_y}\operatorname{arctg}\left(\frac{q_x}{q_y}\right)\pmod{\pi}.
\end{aligned}$$

In the calculation above, we used the transformation rule in Eq.(88) to represent the expression in term of the first Jacobi theta function. Then, by using the logarithm product rule, we assured that the principal value of the infinite sum converge into a function that is well defined in $\mathbb{R}/2\pi\mathbb{Z}$, as discussed in Appx.(5.1). Our last step was to use the identities Eq.(89-90) in order to represent the results as an imaginary part of a compact expression.

In the derivation of the toroidal argument function, we performed the sum over n and only then over m. In other words, we first "folded" the plane to a cylinder about the x-axis and then "folded" the cylinder to a torus about the y-axis. We now describe the result of "folding" the plane about the y-axis first instead. Naively, one may expect that it would merely be equivalent exchanging between the two sides of the rectangle, $q_x \leftrightarrow q_y$ but this shows up to be not accurate. Summing first over m and then over n, when calculating $\phi^T(z, q_x, q_y)$, would yield

$$\lim_{A\to\infty}\sum_{n=-2A}^{2A}\left(\sum_{m=-2A}^{2A}\mathrm{Arg}\left(z+mq_x+inq_y\right)\right)=\mathrm{Im}\left[\mathrm{Log}\left(i\vartheta_1\left(\frac{z}{q_x},\frac{iq_y}{q_x}\right)\right)+\frac{2z^2}{q_xq_y}\mathrm{arctg}\left(\frac{q_y}{q_x}\right)\right]-\frac{\pi}{2}. \tag{27}$$

A comparison of Eq.(27) with Eq.(15) reveals that not only q_x and q_y are exchanged, there is also a rotation of the coordinate system by $\frac{\pi}{2}$ counter-clockwise, $z \longrightarrow e^{i\frac{\pi}{2}}z$.

Derivation of the toroidal argument function when folding order is reversed. We expect that reversing the order that a plane is folded into a torus would result an exchange between q_x and q_y. Thus, we use the identities in Eq.(89-91) to establish the following relation:

$$\vartheta_1\left(\frac{x+iy}{q_x},\frac{iq_y}{q_x}\right)=i\sqrt{\frac{q_x}{q_y}}e^{\pi\frac{q_y}{q_x}\left(\frac{-ix+y}{q_y}\right)^2}\vartheta_1\left(\frac{-ix+y}{q_y},\frac{iq_x}{q_y}\right).$$

Next, we manipulate the relation and obtain that

$$\begin{aligned}&\mathrm{Im}\left[\mathrm{Log}\left(i\vartheta_1\left(\frac{-ix+y}{q_y},\frac{iq_x}{q_y}\right)\right)\right]=\mathrm{Im}\left[\mathrm{Log}\left(i^2\sqrt{\frac{q_y}{q_x}}e^{-\pi\frac{q_y}{q_x}\left(\frac{-ix+y}{q_y}\right)^2}\vartheta_1\left(\frac{x+iy}{q_x},\frac{iq_y}{q_x}\right)\right)\right]=\\&\mathrm{Im}\left[\mathrm{Log}\left(i\vartheta_1\left(\frac{x+iy}{q_x},\frac{iq_y}{q_x}\right)\right)\right]+\frac{2\pi xy}{q_xq_y}-\frac{\pi}{2}\pmod{2\pi}.\end{aligned} \tag{28}$$

We insert it into Eq.(15) together with the identity $\mathrm{arctg}\left(\frac{q_x}{q_y}\right)=\frac{\pi}{2}-\mathrm{arctg}\left(\frac{q_y}{q_x}\right)$ to complete the derivation.

As the cylindrical argument function, the toroidal argument function $\phi^T\left(z, q_x, -iq_y\right)$ also does not share the same periodicity as the torus T, .i.e, it is not Λ-periodic, where $\Lambda = q_x\mathbb{Z} + iq_y\mathbb{Z}$ and $q_x, q_y > 0$. However, it serves as a basic building block for constructing phase fields that are well defined on T. Given a singularity configuration,

$$\Theta^T(z) \equiv \sum_{i=0}^{m-1} k_i\phi^T\left(z - z_i, q_x, -iq_y\right). \tag{29}$$

where $z_i \in \mathbb{C}$ and k_i are the displacement and class of the i^{th} singularity, respectively. For any flat right-angled torus of the form $\Lambda = q_x\mathbb{Z} + iq_y\mathbb{Z}$, the multi-singularity configuration $\Theta^T(\mathbf{r})$ is Λ-periodic iff

$$\sum_{i=0}^{m-1} k_i = \sum_{i=0}^{m-1} k_i z_i = 0. \tag{30}$$

So far we discussed argument functions and multi-vortex configurations on a right-angled torus. In certain instances it is beneficial to describe them on a acute-angled torus, i.e. a flat torus of the form $\mathbb{C}/(\mathbb{Z}\tau_1 + \mathbb{Z}\tau_2)$. Rewriting Eq.(15) with a change of variables, $q_x = \tau_1$ and $q_y = -i\tau_2$, yields a toroidal argument function that is suitable for any flat torus:

$$\phi^T(z, \tau_1, \tau_2) \equiv \mathcal{P.V.} \lim_{A\to\infty} \sum_{m,n=-2A}^{2A} \operatorname{Arg}(z + m\tau_1 + n\tau_2) = \operatorname{Im}\left[\operatorname{Log}\left(i\vartheta_1\left(-\frac{z}{\tau_2}, \frac{\tau_1}{\tau_2}\right)\right) - \frac{2iz^2}{\tau_1\tau_2}\operatorname{arctg}\left(\frac{i\tau_1}{\tau_2}\right)\right], \tag{31}$$

where $\tau_1, \tau_2 \in \mathbb{C}$ span a parallelogram and satisfy $\operatorname{Im}(\bar{\tau}_1\tau_2) > 0$ (since the first Jacobi theta function converges only when $-\tau_1/\tau_2$ is confined to the upper plane). Furthermore, $\operatorname{arctg}(z)$ is defined on the whole complex plane by an analytical continuation (see Appx.(5.2)).

The name "toroidal argument function" is justified by the fact that on one hand it is a singly periodic function and on the other hand it approaches the argument function in a vicinity of a vortex, $\phi^T\left(re^{i\phi}, \tau\right) \xrightarrow[r\to 0]{} \phi$.

We note that τ_2 is a mediator of scalings and rotations on the torus. Any combination of such operations can be viewed as multiplying z by $re^{i\theta} \in \mathbb{C}^\times$ (which yields a scaling by $r > 0$ and a rotation by an angle of θ). We note that it holds that

$$\phi^T\left(re^{i\theta}z, \tau_1, \tau_2\right) = \phi^T\left(z, \frac{1}{r}e^{-i\theta}\tau_1, \frac{1}{r}e^{-i\theta}\tau_2\right). \tag{32}$$

Namely, rotation of z by θ and scaling by r is equivalent to a rotation of τ_1 and τ_2 by $-\theta$ and a scaling by $\frac{1}{r}$.

The generalized toroidal argument function $\phi^T(z)$, which itself is not Λ-periodic, serves as a basic building block for constructions of multi-singularity configurations, $\Theta^T(z)$ that are Λ-periodic. Given a flat torus of the form $\Lambda = \mathbb{Z}\tau_1 + \mathbb{Z}\tau_2$ and a singularity configuration,

$$\Theta^T(\mathbf{r}) \equiv \sum_{i=0}^{m-1} k_i\phi^T(z - z_i, \tau_1, \tau_2). \tag{33}$$

where $z_i \in \mathbb{C}$ and n_i are the displacement and class of the i^{th} singularity, respectively. $\Theta^T(\mathbf{r})$ is Λ-periodic iff

$$\sum_{i=0}^{m-1} k_i = \sum_{i=0}^{m-1} k_i z_i. \tag{34}$$

Derivation of the generalized selection rule. We begin with examining how $\phi^T(z, \tau_1, \tau_2)$ transforms under $z \longrightarrow z + \tau_2$:

$$\begin{aligned}\phi^T(z + \tau_2, \tau_1, \tau_2) &= \operatorname{Im}\left[\operatorname{Log}\left(i\vartheta_1\left(\frac{z}{\tau_2}+1, -\frac{\tau_1}{\tau_2}\right)\right) - \frac{2i(z+\tau_2)^2}{\tau_1\tau_2}\operatorname{arctg}\frac{i\tau_1}{\tau_2}\right] = \\ &\phi^T(z, \tau_1, \tau_2) + \pi - \operatorname{Im}\left[\frac{2i(2z+\tau_2)}{\tau_1}\operatorname{arctg}\frac{i\tau_1}{\tau_2}\right] \pmod{2\pi}.\end{aligned}$$

Next, we use the result to analyze how a multi-vortex configuration, $\Theta^T(z, \tau_1, \tau_2)$ transforms under $z \longrightarrow z + \tau_2$:

$$\Theta^T(z+\tau_2, \tau_1, \tau_2) = \Theta^T(z, \tau_1, \tau_2) + \pi\sum_{i=0}^{m-1} k_i \tag{35}$$

$$- \operatorname{Im}\left[\left(\frac{4i}{\tau_1}\sum_{i=0}^{m-1} k_i(z - z_i) + \frac{2i\tau_2}{\tau_1}\sum_{i=0}^{m-1} k_i\right)\operatorname{arctg}\frac{i\tau_1}{\tau_2}\right] \pmod{2\pi}. \tag{36}$$

Demanding $\Theta^T(z, \tau_1, \tau_2)$ to be τ_2 periodic for any $z \in \mathbb{C}$ leads to the following requirements:

$$\begin{cases} \sum_{i=0}^{m-1} k_i = 0, \\ \operatorname{Im}\left[\frac{4i}{\tau_1}\operatorname{arctg}\left(\frac{i\tau_1}{\tau_2}\right)\sum_{i=0}^{m-1} k_i z_i\right] = 0 \pmod{2\pi}. \end{cases} \tag{37}$$

We now turn to examining how $\phi^T(z, \tau_1, \tau_2)$ transforms under $z \to z + \tau_1$,

$$\phi^T(z+\tau_1, \tau_1, \tau_2) = \operatorname{Im}\left[\operatorname{Log}\left(i\vartheta_1\left(\frac{z+\tau_1}{\tau_2}, -\frac{\tau_1}{\tau_2}\right)\right) - \frac{2i(z+\tau_1)^2}{\tau_1\tau_2}\operatorname{arctg}\frac{i\tau_1}{\tau_2}\right] \pmod{2\pi}.$$

By using the identity in Eq.(92), we check how $\vartheta_1\left(\frac{z}{\tau_2}+\frac{\tau_1}{\tau_2}, -\frac{\tau_1}{\tau_2}\right)$ transforms,

$$\vartheta_1(z-\tau, \tau) = e^{-\pi i\tau + 2\pi i z + \pi i}\vartheta_1(z, \tau) \quad \Rightarrow \quad \vartheta_1\left(\frac{z}{\tau_2} + \frac{\tau_1}{\tau_2}, -\frac{\tau_1}{\tau_2}\right) = e^{i\pi\frac{2z+\tau_1+1}{\tau_2}}\vartheta_1\left(\frac{z}{\tau_2}, -\frac{\tau_1}{\tau_2}\right),$$

and together with

$$\operatorname{Im}\left[\operatorname{Log} e^{i\pi\frac{2z+\tau_1+1}{\tau_2}}\right] = \operatorname{Im}\left[i\pi\frac{2z+\tau_1+1}{\tau_2}\right] \pmod{2\pi},$$

we find that

$$\phi^T(z+\tau_1, \tau_1, \tau_2) = \phi^T(z, \tau_1, \tau_2) + \operatorname{Im}\left[\pi\frac{2z+\tau_1+1}{\tau_2} - \frac{4iz+2i\tau_1}{\tau_2}\operatorname{arctg}\left(\frac{i\tau_1}{\tau_2}\right)\right] \pmod{2\pi}.$$

Our next step is to analyze how a multi-vortex configuration, $\Theta^T(z, \tau_1, \tau_2)$ transforms under $z \to z + \tau_1$.

$$\begin{aligned} \Theta^T(z+\tau_1, \tau_1, \tau_2) = \Theta^T(z, \tau_1, \tau_2) + \operatorname{Im}\Bigg[&\left(\frac{i\tau_1}{\tau_2}\left(\pi - 2\operatorname{arctg}\frac{i\tau_1}{\tau_2}\right) + i\pi\right)\sum_{i=0}^{m-1} k_i + \\ &+\frac{2i}{\tau_2}\left(\pi - 2\operatorname{arctg}\frac{i\tau_1}{\tau_2}\right)\sum_{i=0}^{m-1} k_i(z - z_i)\Bigg] \pmod{2\pi} \end{aligned} \tag{38}$$

Demanding $\Theta^T(z, \tau_1, \tau_2)$ to be τ_1 periodic for any $z \in \mathbb{C}$ leads to the following requirements:

$$\begin{cases} \sum_{i=0}^{m-1} k_i = 0, \\ \operatorname{Im}\left[\frac{2i}{\tau_2}\left(\pi - 2\operatorname{arctg}\frac{i\tau_1}{\tau_2}\right)\sum_{i=0}^{m-1} k_i z_i\right] = 0 \pmod{2\pi}. \end{cases} \tag{39}$$

Combining the conditions above gives a selection rules,

$$\begin{cases} \sum_{i=0}^{m-1} k_i = 0, \\ \operatorname{Im}\left[\frac{4i}{\tau_1}\operatorname{arctg}\left(\frac{i\tau_1}{\tau_2}\right)\sum_{i=0}^{m-1} k_i z_i\right] \equiv 0 \pmod{2\pi}, \\ \operatorname{Im}\left[\frac{2i}{\tau_2}\left(\pi + 2\operatorname{arctg}\left(\frac{i\tau_1}{\tau_2}\right)\right)\sum_{i=0}^{m-1} k_i z_i\right] \equiv 0 \pmod{2\pi}, \end{cases} \tag{40}$$

which determine whether a multi-vortex configuration, $\phi^T(z,\tau)$ is Λ-periodic. The configuration would be periodic with respect to any flat torus, regardless to its circumferences, iff

$$\sum_{i=0}^{m-1} k_i = \sum_{i=0}^{m-1} k_i z_i, \tag{41}$$

which coincide we the selection rule for a right-angled flat torus.

Next, we consider the multi-vortex configuration,

$$\Theta^T(z) = \sum_{k=0}^{3}(-1)^i \phi^T(z - z_i, \tau_1, \tau_2) \tag{42}$$

where two vortices with winding number +1 are located at opposite corners of a parallelogram and the second pair of vortices with the opposite winding number are located at the other opposing corners. The selection rule reveals that this multi-vortex configuration is Λ-periodic regardless of the torus dimensions. In addition, it is the minimal configuration of vortices with a single winding which is Λ-periodic.

The selection rule allow a parallelogram to spread over a few tiles. In addition, a vortex that is placed in one tile would also appear in all the other replicas. Hence, the selection rule is actually less restrictive then it seems at first sight, making it for applicable to many scenarios.

3 Placing a single V-AV pair on a torus

One might hold to the notion that it might not be possible to place a single V-AV pair on a flat torus. However we would like to suggest evidence to the contrary. The toroidal argument function consists of two terms. The first term encodes the singularities completely, while the second one only enforces the selection rules. Furthermore, for supported multi-singularity configurations the second term amounts to some constant, which donates only to some globally fixed phase.

In other words you would suspect this second term can be eliminated with extreme prejudice. This is not completely true, since the periodicity still need to be enforced on the torus. Thus we are free to replace this second term with an equivalent term, that enforces them. A simple ansatz that recovers the single-valuedness (mod 2π) of the phase field is $\Theta^{T'}(z) = \phi^{T'}(z, w_0) - \phi^{T'}(z, w_1)$ with

$$\phi^{T'}(z,w) \equiv \mathrm{Im}\left[\mathrm{Log}\left(i\vartheta_1\left(\frac{z-w}{\tau_2}, -\frac{\tau_1}{\tau_2}\right)\right) + \frac{2\pi z}{\tau_2}\frac{\mathrm{Re}(w/\tau_2)}{\mathrm{Im}(\tau_1/\tau_2)}\right], \tag{43}$$

where the second term ensures it is single-valued (mod 2π). Any multi-singularity configuration that obey the selection rule in Eq.(41) is the same (up to constant) regardless to the building block that was used.

Asserting the periodicity of the minimal singularity configuration.

$$\Theta^{T'}(z+\tau_1) = \mathrm{Im}\left[\mathrm{Log}\left(\vartheta_1\left(\frac{z-w_0}{\tau_2} + \frac{\tau_1}{\tau_2}\right)\bar{\vartheta}_1\left(\frac{z-w_1}{\tau_2} + \frac{\tau_1}{\tau_2}\right)\right)\right] + \frac{2\pi(z+\tau_1)}{\tau_2}\frac{\mathrm{Re}(\frac{w_0-w_1}{\tau_2})}{\mathrm{Im}(\frac{\tau_1}{\tau_2})} =$$

$$\Theta^{T'}(z) + \mathrm{Im}\left[2\pi i\left(\frac{z-w_0}{\tau_2} - \frac{z-w_1}{\tau_2}\right) + \frac{2\pi\tau_1}{\tau_2}\frac{\mathrm{Re}(\frac{w_0-w_1}{\tau_2})}{\mathrm{Im}(\frac{\tau_1}{\tau_2})}\right] = \Theta^{T'}(z) \pmod{2\pi}$$

where in the second step we used the relation $\vartheta_1(z-\tau,\tau)\bar{\vartheta}_1(z'-\tau,\tau)=\vartheta_1(z,\tau)\bar{\vartheta}_1(z',\tau)e^{i2\pi(z-z')}$.

$$\Theta^{T'}(z+\tau_2)=\Theta^{T'}(z)-\mathrm{Im}\left(\frac{\tau_2}{\tau_2}\right)\frac{\mathrm{Re}(\frac{w_0-w_1}{\tau_2})}{\mathrm{Im}(\frac{\tau_1}{\tau_2})}=\Theta^{T'}(z)\pmod{2\pi} \tag{44}$$

where in the second step we used the relation $\vartheta_1(z+1,\tau)\bar{\vartheta}_1(z'+1,\tau)=\vartheta_1(z,\tau)\bar{\vartheta}_1(z',\tau)$.

We would like to compare between the toroidal argument functions appearing in Eq.(31 and Eq.(43). A calculation of the supercurrent flux through the parallelogram sides, τ_1 and τ_2 (or any parallel dissection) for a V-AV yields:

$$\phi(\tau)=\int_\tau \mathrm{d}z\,\mathbf{J}(z)\cdot(\hat{z}\times\hat{\tau})=\begin{cases}\frac{2\pi}{\mathrm{Im}(\tau_1/\tau_2)}\mathrm{Re}\left(\frac{w_0-w_1}{\tau_2}\right), & \tau=\tau_2\\ 2\pi\left[\mathrm{Im}\left(\frac{w_0-w_1}{\tau_2}\right)+\mathrm{Re}\left(\frac{w_0-w_1}{\tau_2}\right)\frac{\mathrm{Re}(\tau_1/\tau_2)}{\mathrm{Im}(\tau_1/\tau_2)}\right], & \tau=\tau_1\end{cases}. \tag{45}$$

This calculation reveals that multi-vortex configuration satisfying the selection rule in Eq.(34) do not produce super-current flux through the parallelogram sides.

Calculation of the supercurrent flux through the parallelogram sides. We start with the first term ,which encode the singularities:

$$\begin{aligned}&\int_{t=0}^{t=1}\mathrm{d}t\,|\mathbf{R}'(t)|\nabla_{\mathbf{R}}\mathrm{Im}\left[\mathrm{Log}\left(i\vartheta_1\left(\frac{z(\mathbf{R}(t))}{\tau_2}\right)\right)\right]\cdot(\hat{z}\times\hat{\tau})=\\&\qquad\mathrm{Im}\left[\int_0^1\mathrm{d}t\,|\tau|\,\nabla_{\mathbf{R}}\mathrm{Log}\left(i\vartheta_1\left(\frac{z(\mathbf{R}(t))}{\tau_2}\right)\right)\cdot\frac{(-\mathrm{Im}\tau,\mathrm{Re}\tau)}{|\tau|}\right]=\\&\mathrm{Im}\left[\int_0^1\mathrm{d}t\,\partial_z\mathrm{Log}\left(i\vartheta_1\left(\frac{z(t)}{\tau_2}\right)\right)(1,i)\cdot(-\mathrm{Im}\tau,\mathrm{Re}\tau)\right]=\mathrm{Im}\left[\int_0^1\mathrm{d}t\,\partial_z\mathrm{Log}\left(i\vartheta_1\left(\frac{z(t)}{\tau_2}\right)\right)i\tau\right]\end{aligned} \tag{46}$$

$$=\left\{\begin{matrix}z=z_0+\tau t\\ \Rightarrow\mathrm{d}z=\tau\mathrm{d}t\end{matrix}\right\}=\mathrm{Re}\left[\int_{z_0}^{z_0+\tau}\mathrm{d}z\,\partial_z\mathrm{Log}\left(\vartheta_1\left(\frac{z}{\tau_2},-\frac{\tau_1}{\tau_2}\right)\right)\right]=\begin{cases}0, & \tau=\tau_2\\ -\pi\mathrm{Im}\left(\frac{2z_0+\tau_1}{\tau_2}\right), & \tau=\tau_1\end{cases}, \tag{47}$$

where $\mathbf{R}=(\mathrm{Re}(z),\mathrm{Im}(z))$, $z(t)=z_0+\tau t$ and $\hat{\tau}=(\mathrm{Re}(\tau),\mathrm{Im}(\tau))$. Therefore, for a single V-AV pair the super-current flux through a line parallel to the parallelogram side τ_i is

$$\phi_a(\tau)=\int_{t=0}^{t=1}\mathrm{d}t\,|\mathbf{R}'(t)|\,\mathbf{J}_a(\mathbf{R}(t))\cdot(\hat{z}\times\hat{\tau})=\begin{cases}0, & \tau=\tau_2\\ 2\pi\mathrm{Im}\left(\frac{w_0-w_1}{\tau_2}\right), & \tau=\tau_1\end{cases} \tag{48}$$

where the super-current density is

$$\mathbf{J}_a(\mathbf{R})=\nabla_{\mathbf{R}}\mathrm{Im}\left[\mathrm{Log}\left(\vartheta_1\left(\frac{z(\mathbf{R})-w_0}{\tau_2}\right)\bar{\vartheta}_1\left(\frac{z(\mathbf{R})-w_1}{\tau_2}\right)\right)\right] \tag{49}$$

Now we turn to analyze the second term which is $\propto \mathrm{Im}(z/\tau_2)$:

$$\mathrm{Im}\left(\frac{1}{\tau_2}\int_0^1 \mathrm{d}t\ |\mathbf{R}'(t)|\ \nabla_{\mathbf{R}} z(\mathbf{R}(t))\cdot(\hat{z}\times\hat{\tau})\right) = \mathrm{Im}\left(\frac{1}{\tau_2}\int_0^1 \mathrm{d}t\ |\tau|\ \frac{(1,i)\cdot(-\mathrm{Im}\tau,\mathrm{Re}\tau)}{|\tau|}\right)$$
$$= \mathrm{Im}\left[\frac{1}{\tau_2}\int_0^1 \mathrm{d}t\ i\tau\right] = \mathrm{Re}\left(\frac{\tau}{\tau_2}\right) \tag{50}$$

For a single V-AV pair the super-current flux through a line parallel to the parallelogram side τ_i is

$$\phi_b(\tau_i) = \int_{t=0}^{t=1} \mathrm{d}t\ |\mathbf{R}'(t)|\ \mathbf{J}_b(\mathbf{R}(t))\cdot(\hat{z}\times\hat{\tau}) = 2\pi\mathrm{Re}\left(\frac{w_0-w_1}{\tau_2}\right)\frac{\mathrm{Re}(\tau_i/\tau_2)}{\mathrm{Im}(\tau_1/\tau_2)} \tag{51}$$

where the super-current density is

$$\mathbf{J}_b(\mathbf{R}) = 2\pi\nabla_{\mathbf{R}}\mathrm{Im}\left(\frac{z(\mathbf{R})}{\tau_2}\right)\frac{\mathrm{Re}(\frac{w_0-w_1}{\tau_2})}{\mathrm{Im}(\frac{\tau_1}{\tau_2})} \tag{52}$$

4 Superconducting flat torus

We suggest a general multi-vortex configuration takes the form

$$\Theta^L(z) = \sum_i k_i\phi^L(z,z_i), \tag{53}$$

where $k_i \in \mathbb{Z}$ is the winding number that characterize the i vortex and z_i is its position. The fundamental building block, which differs by a smooth function from the toroidal argument function, is given by

$$\begin{aligned}\phi^L(z,z_0) = \mathrm{Im}\left[\mathrm{Log}\left(i\vartheta_1\left(\frac{z-z_0}{\tau_2},-\frac{\tau_1}{\tau_2}\right)\right)\right] + q_1\pi\,\mathrm{Re}\left(\frac{z^2}{\tau_2\tau_1}\right) - q_2\pi\frac{\mathrm{Im}^2(z/\tau_2)\,\mathrm{Re}(\tau_1/\tau_2)}{\mathrm{Im}^2(\tau_1/\tau_2)}\\ - q_1\pi\frac{\mathrm{Im}^2(z/\tau_1)\,\mathrm{Re}(\tau_2/\tau_1)}{\mathrm{Im}^2(\tau_2/\tau_1)} + \pi\frac{\mathrm{Im}(z/\tau_1)}{\mathrm{Im}(\tau_2/\tau_1)} + \left[2\pi\,\mathrm{Re}\left(\frac{z_0}{\tau_2}\right)-\pi\right]\frac{\mathrm{Im}(z/\tau_2)}{\mathrm{Im}(\tau_1/\tau_2)},\end{aligned} \tag{54}$$

where q_i represents the number of lattice sites along τ_i and $q_2 = q_1 + 1$, which is necessary in order to maintain the single-valuedness (mod 2π) of $\phi^L(z,z_0)$ at every lattice point, $z_{m,n} = (m/q_1)\tau_1 + (n/q_2)\tau_2$ with $m,n \in \mathbb{Z}$ - i.e., $\phi^L(z_{m,n}+\tau_i,z_0) - \phi^L(z_{m,n},z_0) = 0$ (mod 2π). Furthermore, integrating the supercurrent, $\mathbf{J} \propto \left(\frac{1}{2}\boldsymbol{\nabla}\phi^L - \boldsymbol{A}\right)$ along the parallelogram, spanned by τ_1 and τ_2 reveals that

$$0 = \oint \mathbf{J}\cdot\mathbf{d}\boldsymbol{\ell} \propto N\Phi_0 - \oint \mathbf{A}\cdot\mathbf{d}\boldsymbol{\ell}, \tag{55}$$

and due to Dirac quantization condition ($\frac{e_M e}{2\pi\epsilon_0\hbar c^2} \in \mathbb{Z}$), $N = \sum_i k_i$ must be an even number.

Proving that $\phi^L(z, z_0)$ is single valued (mod 2π) at the lattice sites. We start by examining how each one of the first three terms in $\phi^L(z, z_0)$ changes as we circulate around the torus holes:

$$\begin{aligned}
&\vartheta_1(z + a + b\tau, \tau) = \exp\left[i\pi(a - b(2z + b\tau + 1)\right]\vartheta_1(z,\tau) \Longrightarrow \vartheta_1\left(\frac{z - z_0 + a\tau_2 + b\tau_1}{\tau_2}, -\frac{\tau_1}{\tau_2}\right) = \\
&\quad \exp\left[i\pi\left(a + b\left(2\frac{z - z_0}{\tau_2} + b\frac{\tau_1}{\tau_2} + 1\right)\right)\right]\vartheta_1(z,\tau) \\
&\Longrightarrow \operatorname{Im}\left[\operatorname{Log}\left(\vartheta_1\left(\frac{z - z_0 + a\tau_2 + b\tau_1}{\tau_2}, -\frac{\tau_1}{\tau_2}\right)\right)\right] - \operatorname{Im}\left[\operatorname{Log}\left(\vartheta_1\left(\frac{z - z_0}{\tau_2}, -\frac{\tau_1}{\tau_2}\right)\right)\right] = \\
&\quad \pi(a - b) + 2\pi b \operatorname{Re}\left(\frac{z - z_0}{\tau_2}\right) + \pi b^2 \operatorname{Re}\left(\frac{\tau_1}{\tau_2}\right)
\end{aligned} \tag{56}$$

$$\operatorname{Re}\left(\frac{(z + b\tau_1)^2}{\tau_2\tau_1}\right) - \operatorname{Re}\left(\frac{z^2}{\tau_2\tau_1}\right) = 2b\operatorname{Re}\left(\frac{z\tau_1}{\tau_2\tau_1}\right) + b^2\operatorname{Re}\left(\frac{\tau_1^2}{\tau_2\tau_1}\right) = 2b\operatorname{Re}\left(\frac{z}{\tau_2}\right) + b^2\operatorname{Re}\left(\frac{\tau_1}{\tau_2}\right) \tag{57}$$

$$\operatorname{Re}\left(\frac{(z + a\tau_2)^2}{\tau_2\tau_1}\right) - \operatorname{Re}\left(\frac{z^2}{\tau_2\tau_1}\right) = 2a\operatorname{Re}\left(\frac{z\tau_2}{\tau_2\tau_1}\right) + a^2\operatorname{Re}\left(\frac{\tau_2^2}{\tau_2\tau_1}\right) = 2a\operatorname{Re}\left(\frac{z}{\tau_1}\right) + a^2\operatorname{Re}\left(\frac{\tau_2}{\tau_1}\right) \tag{58}$$

$$\begin{aligned}
&\operatorname{Im}^2\left(\frac{z + b\tau_1}{\tau_2}\right) = \left[\operatorname{Im}\left(\frac{z}{\tau_2}\right) + b\operatorname{Im}\left(\frac{\tau_1}{\tau_2}\right)\right]^2 = \operatorname{Im}^2\left(\frac{z}{\tau_2}\right) + 2b\operatorname{Im}\left(\frac{z}{\tau_2}\right)\operatorname{Im}\left(\frac{\tau_1}{\tau_2}\right) + b^2\operatorname{Im}^2\left(\frac{\tau_1}{\tau_2}\right) \\
&\quad \Longrightarrow \frac{\operatorname{Re}(\tau_1/\tau_2)}{\operatorname{Im}^2(\tau_1/\tau_2)}\left[\operatorname{Im}^2\left(\frac{z + b\tau_1}{\tau_2}\right) - \operatorname{Im}^2\left(\frac{z}{\tau_2}\right)\right] = \frac{2b\operatorname{Im}(z/\tau_2)\operatorname{Re}(\tau_1/\tau_2)}{\operatorname{Im}(\tau_1/\tau_2)} + b^2\operatorname{Re}(\tau_1/\tau_2)
\end{aligned} \tag{59}$$

Thus, the function $\phi(z, z_0)$ changes by

$$\begin{aligned}
&\phi^L(z + b\tau_1) - \phi^L(z) = (q_1 + 1)2\pi b\operatorname{Re}\left(\frac{z}{\tau_2}\right) + \pi b^2(q_1 + 1)\operatorname{Re}\left(\frac{\tau_1}{\tau_2}\right) - \pi q_2 b^2\operatorname{Re}\left(\frac{\tau_1}{\tau_2}\right) \\
&\quad - \frac{2\pi q_2 b\operatorname{Im}(z/\tau_2)\operatorname{Re}(\tau_1/\tau_2)}{\operatorname{Im}(\tau_1/\tau_2)} \pmod{2\pi},
\end{aligned} \tag{60}$$

as we circulate along τ_1. For the case that $q_2 = q_1 + 1$ we find that $\phi^L(z, z_0)$ is single-valued (mod 2π) at the lattice sites, $z_{m,n} = (m/q_1)\tau_1 + (n/q_2)\tau_2$ with $m, n \in \mathbb{Z}$,

$$\begin{aligned}
&\phi^L(z_{mn} + b\tau_1) - \phi^L(z_{mn}) = 2\pi q_2 b\left[\operatorname{Re}\left(\frac{z_{mn}}{\tau_2}\right) - \frac{\operatorname{Re}(\tau_1/\tau_2)}{\operatorname{Im}(\tau_1/\tau_2)}\operatorname{Im}\left(\frac{z_{mn}}{\tau_2}\right)\right] \pmod{2\pi} \\
&= 2\pi nb \pmod{2\pi} = 0 \pmod{2\pi}.
\end{aligned} \tag{61}$$

In addition, the function $\phi(z, z_0)$ changes by

$$\begin{aligned}
&\phi^L(z_{mn} + a\tau_2) - \phi^L(z_{mn}) = +2\pi q_1 a\left[\operatorname{Re}\left(\frac{z_{mn}}{\tau_1}\right) - \frac{\operatorname{Re}(\tau_2/\tau_1)}{\operatorname{Im}(\tau_2/\tau_1)}\operatorname{Im}\left(\frac{z_{mn}}{\tau_1}\right)\right] \pmod{2\pi} \\
&= 2\pi ma \pmod{2\pi} = 0 \pmod{2\pi}
\end{aligned} \tag{62}$$

as we circulate along τ_2 and therefore single valued (mod 2π) at the entire space.

Next, we introduce a complementary vector potential. Due to the boundary conditions, the supercurrent is required to be doubly periodic, $\mathbf{J}(z+\tau_i)-\mathbf{J}(z)=0$. In addition, we want a homogeneous magnetic field, $\boldsymbol{\nabla}\times\mathbf{A}=const$. Thus, the complementary vector field needs to fulfill the condition

$$\mathbf{A}(z+\frac{\tau_i}{q_i})=\mathbf{A}(z)+\frac{p\Phi_0}{\pi q_i}\boldsymbol{\nabla}\chi_i, \tag{63}$$

where $\chi_i(z)\equiv\phi^L(z+\tau_i)-\phi^L(z)$ for $i=1,2$, $p=\lfloor N/2\rfloor$ and $\boldsymbol{\nabla}\chi_i=const$. In addition, we take $\hbar=e=c=1$ and thus $\Phi_0=hc/(2e)=\pi$.

The condition in Eq.(63) assures us the following:

1. The flux through each unit cell is quantized,

$$\oiint \mathrm{d}\mathbf{S}(\boldsymbol{\nabla}\times\mathbf{A})=\frac{2p\Phi_0}{q_1(q_1+1)}. \tag{64}$$

Proof:

$$\begin{aligned}&\oint \mathbf{A}\cdot\mathrm{d}\boldsymbol{\ell}=\int_{z_{m,n}}^{z_{m,n}+\frac{\tau_2}{q_2}}\left[\mathbf{A}(z+\frac{\tau_1}{q_1})-\mathbf{A}(z)\right]\cdot\mathrm{d}\boldsymbol{\ell}+\int_{z_{m,n}}^{z_{m,n}+\frac{\tau_1}{q_1}}\left[\mathbf{A}(z)-\mathbf{A}(z+\frac{\tau_2}{q_2})\right]\cdot\mathrm{d}\boldsymbol{\ell}=\\&\frac{p}{q_1}\int_{z_{m,n}}^{z_{m,n}+\frac{\tau_2}{q_2}}\nabla\chi_1\cdot\mathrm{d}\boldsymbol{\ell}-\frac{p}{q_2}\int_{z_{m,n}}^{z_{m,n}+\frac{\tau_1}{q_1}}\boldsymbol{\nabla}\chi_2\cdot\mathrm{d}\boldsymbol{\ell}=\frac{p}{q_1}\chi_1(z)\Big]_{z_{m,n}}^{z_{m,n+1}}-\frac{p}{q_2}\chi_2(z)\Big]_{z_{m,n}}^{z_{m+1,n}}=\\&2\pi p(\frac{1}{q_1}-\frac{1}{q_2})=2\pi p\frac{q_2-q_1}{q_1q_2}=\frac{2\pi p}{q_1(q_1+1)}.\end{aligned} \tag{65}$$

2. The phase factors in the Peierls substitution method are single valued:

$$\begin{aligned}&\int_{z_{m,n}}^{z_{m,n}+\frac{\tau_2}{q_2}}[\mathbf{A}(z+\tau_1)-\mathbf{A}(z)]\cdot\mathrm{d}\boldsymbol{\ell}=0\pmod{2\pi}\\&\int_{z_{m,n}}^{z_{m,n}+\frac{\tau_1}{q_1}}[\mathbf{A}(z+\tau_2)-\mathbf{A}(z)]\cdot\mathrm{d}\boldsymbol{\ell}=0\pmod{2\pi}\end{aligned} \tag{66}$$

3. The flux through the torus surface is quantized, $\oiint \mathrm{d}\mathbf{S}(\boldsymbol{\nabla}\times\mathbf{A})=2p\Phi_0$.

Proof:

$$\begin{aligned}&\oint \mathbf{A}\cdot\mathrm{d}\boldsymbol{\ell}=\int_{z_{m,n}}^{z_{m,n}+\tau_2}[\mathbf{A}(z+\tau_1)-\mathbf{A}(z)]\cdot\mathrm{d}\boldsymbol{\ell}+\int_{z_{m,n}}^{z_{m,n}+\tau_1}[\mathbf{A}(z)-\mathbf{A}(z+\tau_2)]\cdot\mathrm{d}\boldsymbol{\ell}=\\&p\int_{z_{m,n}}^{z_{m,n}+\tau_2}\nabla\chi_1\cdot\mathrm{d}\boldsymbol{\ell}-p\int_{z_{m,n}}^{z_{m,n}+\tau_1}\nabla\chi_2\cdot\mathrm{d}\boldsymbol{\ell}=2\pi p(q_2-q_1)=2\pi p.\end{aligned} \tag{67}$$

4. When the winding number around the magnetic unit cell is even, a translation by a lattice vector, τ_i would amount to applying a gauge transformation,

$$\Psi(z)=\mathbf{M}_i(z)\Psi(z+\tau_i), \tag{68}$$

where $\Psi^\dagger(z)=(\psi^\dagger(z),\psi(z))$ is a particle-hole spinor and $\mathbf{M}_i(z)=e^{-i\sigma_z\chi_i(z)/2}$ are translation operators which commute with each other, $[\mathbf{M}_1,\mathbf{M}_2]=0$.

We notice that $\chi_2(z)$ can be obtained from $\chi_1(z)$ by exchanging the indexes $1 \longleftrightarrow 2$.

$$
\begin{aligned}
\nabla\chi_1(z) &= 2\pi q_2 \left[\mathrm{Re}\left(\frac{\hat{x}+i\hat{y}}{\tau_2}\right) - \frac{\mathrm{Re}\,(\tau_1/\tau_2)}{\mathrm{Im}\,(\tau_1/\tau_2)}\mathrm{Im}\left(\frac{\hat{x}+i\hat{y}}{\tau_2}\right)\right] \\
&= 2\pi q_2 \left[\left(\mathrm{Re}\left(\frac{1}{\tau_2}\right) - \frac{\mathrm{Re}\,(\tau_1/\tau_2)}{\mathrm{Im}\,(\tau_1/\tau_2)}\mathrm{Im}\left(\frac{1}{\tau_2}\right)\right)\hat{x} - \left(\mathrm{Im}\left(\frac{1}{\tau_2}\right) + \frac{\mathrm{Re}\,(\tau_1/\tau_2)}{\mathrm{Im}\,(\tau_1/\tau_2)}\mathrm{Re}\left(\frac{1}{\tau_2}\right)\right)\hat{y}\right] \\
&= \frac{2\pi}{q_2 c_2^2}\left[\left(\mathrm{Re}\,(\tau_2) - \frac{\mathrm{Re}\,(\bar{\tau}_1\tau_2)}{\mathrm{Im}\,(\bar{\tau}_1\tau_2)}\mathrm{Im}\,(\tau_2)\right)\hat{x} + \left(\mathrm{Im}\,(\tau_2) + \frac{\mathrm{Re}\,(\bar{\tau}_1\tau_2)}{\mathrm{Im}\,(\bar{\tau}_1\tau_2)}\mathrm{Re}\,(\tau_2)\right)\hat{y}\right]
\end{aligned} \tag{69}
$$

where in the last equality we used the relations

$$
\begin{aligned}
\mathrm{Re}\left(\frac{q_i}{\tau_i}\right) &= \frac{q_i}{|\tau_i|^2}\mathrm{Re}\,(\bar{\tau}_i) = \frac{1}{q_i c_i^2}\mathrm{Re}\,(\tau_i), \\
\mathrm{Im}\left(\frac{q_i}{\tau_i}\right) &= \frac{q_i}{|\tau_i|^2}\mathrm{Im}\,(\bar{\tau}_i) = -\frac{1}{q_i c_i^2}\mathrm{Im}\,(\tau_i),
\end{aligned} \tag{70}
$$

and $c_i = |\tau_i/q_i|$ is the lattice constant in the τ_i direction. Based on Eq.(63), we construct the vector potential $A(z)$ as follows:

$$
\begin{aligned}
A(z) &= \frac{\Phi_0 p}{\pi}\left(\nabla\chi_2(z)\frac{\mathrm{Im}\,(z/\tau_1)}{\mathrm{Im}\,(\tau_2/\tau_1)} + \nabla\chi_1(z)\frac{\mathrm{Im}\,(z/\tau_2)}{\mathrm{Im}\,(\tau_1/\tau_2)}\right) = \\
&\frac{\Phi_0 p}{\mathrm{Im}\,(\bar{\tau}_1\tau_2)\,\pi}\left(\nabla\chi_2(z)\mathrm{Im}\,(z\bar{\tau}_1) - \nabla\chi_1(z)\mathrm{Im}\,(z\bar{\tau}_2)\right) = \\
&= |\epsilon_{ij}|\frac{2\Phi_0 p\,\mathrm{Im}\,(z\bar{\tau}_i)}{\mathrm{Im}\,(\bar{\tau}_i\tau_j)\,q_i c_i^2}\left[\left(\mathrm{Re}\,(\tau_i) + \frac{\mathrm{Re}\,(\bar{\tau}_i\tau_j)}{\mathrm{Im}\,(\bar{\tau}_i\tau_j)}\mathrm{Im}\,(\tau_i)\right)\hat{x} + \left(\mathrm{Im}\,(\tau_i) - \frac{\mathrm{Re}\,(\bar{\tau}_i\tau_j)}{\mathrm{Im}\,(\bar{\tau}_i\tau_j)}\mathrm{Re}\,(\tau_i)\right)\hat{y}\right],
\end{aligned} \tag{71}
$$

where $i, j \in \{1, 2\}$ and ϵ_{ij} is the Levi-Chivita tensor.

Let us check that indeed the curl of the magnetic vector potential $A(z)$ is as we expected:

$$
\begin{aligned}
&B(z) = (\nabla\times A)_z = \partial_x A_y - \partial_y A_x = \\
&|\epsilon_{ij}|\frac{2\Phi_0 p}{\mathrm{Im}\,(\bar{\tau}_i\tau_j)\,q_i c_i^2}\left[\left(\frac{\mathrm{Re}\,(\bar{\tau}_i\tau_j)}{\mathrm{Im}\,(\bar{\tau}_i\tau_j)}\mathrm{Re}\,(\tau_i) - \mathrm{Im}\,(\tau_i)\right)\mathrm{Im}\,(\tau_i) - \left(\mathrm{Re}\,(\tau_i) + \frac{\mathrm{Re}\,(\bar{\tau}_i\tau_j)}{\mathrm{Im}\,(\bar{\tau}_i\tau_j)}\mathrm{Im}\,(\tau_i)\right)\mathrm{Re}\,(\tau_i)\right] \\
&= -|\epsilon_{ij}|\frac{2\Phi_0 p q_i}{\mathrm{Im}\,(\bar{\tau}_i\tau_j)} = \frac{2\Phi_0 p}{\mathrm{Im}\,(\bar{\tau}_1\tau_2)}(q_2 - q_1) = \frac{2\Phi_0 p}{\mathrm{Im}\,(\bar{\tau}_1\tau_2)}
\end{aligned} \tag{72}
$$

In the simple case of a rectangular lattice with the distance between nearest neighbor lattice sites taken to be unity, $\tau_x = q$ and $\tau_y = i(q+1)$, we have,

$$
\begin{aligned}
\phi^L(x, y, x_0, y_0) &= \mathrm{Im}\left[\mathrm{Log}\left(i\vartheta_1\left(\frac{z - z_0}{i(q+1)}, i\frac{q}{q+1}\right)\right)\right] + \frac{2\pi xy}{q+1} + \frac{2\pi y_0 x}{q+1} + \frac{\pi x}{q} - \frac{\pi y}{q+1} \\
A(x, y) &= 2\Phi_0 p\left(\frac{y}{q+1}\hat{x} + \frac{x}{q}\hat{y}\right)
\end{aligned} \tag{73}
$$

5 Intermezzo

5.1 Lifting of arctg

For $z = x + iy = re^{i\theta} \in \mathbb{C}^{\times}$ the principal value of the logarithm is given by

$$\operatorname{Log} z := \ln r + i\theta, \tag{74}$$

where θ is taken in $(\pi, \pi]$. It can also be written in terms of the principal value of the argument function, $\operatorname{Arg}\colon \mathbb{C}\setminus\{0\} \to \mathbb{R}/2\pi\mathbb{Z}$,

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z. \tag{75}$$

This connection can also be written in terms of the principal value

$$\operatorname{Arg} z = \frac{i}{2} (\operatorname{Log} \overline{z} - \operatorname{Log} z) = \operatorname{Im} [\operatorname{Log} (z)] \,. \tag{76}$$

Hence, the function $\operatorname{atan2}(y, x)$ can be represented in terms of the Argument function,

$$\operatorname{atan2}(y, x) = \operatorname{Arg}(x+iy) = \frac{i}{2}(\operatorname{Log}(x-iy) - \operatorname{Log}(x+iy)) \,. \tag{77}$$

In a similar fashion, we represent the arctangent function as,

$$\operatorname{arctg}\left(\frac{y}{x}\right) = \frac{i}{2} \operatorname{Log}(\frac{x-iy}{x+iy}). \tag{78}$$

Comparing the two representations, we notice that the product formula of the logarithm can be used to lift of $\operatorname{arctg}\left(\frac{y}{x}\right)$ to $\mathbb{R}/2\pi\mathbb{Z}$,

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\operatorname{arctg}\left(\frac{y}{x}\right)} & \mathbb{R}\big/\pi\mathbb{Z} \\ & \searrow^{\operatorname{atan2}(y,x)} & \big\downarrow \text{Log product formula} \\ & & \mathbb{R}\big/2\pi\mathbb{Z} \end{array} \,. \tag{79}$$

This trick is used throughout the article to lift the toroidal and cylindrical argument functions into $\mathbb{R}/2\pi\mathbb{Z}$.

5.2 Analytical continuation of arctg

We take Eq.(78) and substitute $z = y/x$. We then perform an analytical continuation of arctg by extending its domain to the whole complex plane, $z \in \mathbb{C}$,

$$\operatorname{arctg}(z) = \frac{i}{2} \operatorname{Log}\left(\frac{1+iz}{1-iz}\right). \tag{80}$$

6 Special Functions

6.1 Jacobi theta functions

Here, we only introduce the theta functions which are used in the paper together with their relevant properties. There are a number of notational systems for the Jacobi theta functions and it is the notation given in Whittaker and Watson[3, 4, 5] that we adopt.

i. *The third Jacobi theta function*

For any $z, \tau \in \mathbb{C}$ such that $\mathrm{Re}\,\tau > 0$ we define the third Jacobi theta function to be

$$\vartheta_3(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z} = \sum_{n=-\infty}^{\infty} q^{n^2} w^{2n}, \tag{81}$$

where $q = e^{\pi i \tau}$ and $w = e^{\pi i z}$. The third Jacobi theta function satisfies the following properties:

- It is quasi-periodic,

$$\vartheta_3(z + a + b\tau, \tau) = e^{-\pi i b^2 \tau - 2\pi i b z} \vartheta_3(z,\tau) \quad \forall a, b \in \mathbb{Z}. \tag{82}$$

- It is presented by Jacobi triple product formula,

$$\vartheta_3(z,\tau) = \prod_{m=1}^{\infty} \left(1 - q^{2m}\right)\left(1 + w^2 q^{2m-1}\right)\left(1 + w^{-2} q^{2m-1}\right). \tag{83}$$

- It supports the transformation $\tau \to -{}^{1}/_{\tau}$,

$$\vartheta_3\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} e^{\frac{\pi}{\tau} i z^2} \vartheta_3(z,\tau). \tag{84}$$

- Zeros of $\vartheta_3(z,\tau)$ occur when $z \in \left(\mathbb{Z} + \frac{1}{2}\right)\tau + \mathbb{Z} + \frac{1}{2}$. • It is anti-symmetric with respect to z,

$$\vartheta_3(z,\tau) = \vartheta_3(-z,\tau). \tag{85}$$

- It transforms under conjugation as follows,

$$\overline{\vartheta_3(z,\tau)} = \vartheta_3(\bar{z}, -\bar{\tau}). \tag{86}$$

ii. *The first Jacobi theta function*

For any $z, \tau \in \mathbb{C}$ such that $\mathrm{Re}\,\tau > 0$ we define the third Jacobi theta function to be

$$\vartheta_1(z) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2} \exp((2n+1) i \pi z) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2} w^{2n+1}, \tag{87}$$

where $q = e^{\pi i \tau}$ and $w = e^{\pi i z}$. The first Jacobi theta function satisfies the following properties:

- The first and third theta functions are related:

$$\vartheta_1(z,\tau) = e^{\frac{1}{4}\pi i \tau + \pi i \left(z + \frac{3}{2}\right)} \vartheta_3\left(z + \frac{1}{2} + \frac{1}{2}\tau, \tau\right), \tag{88}$$

- It is symmetric with respect to z,

$$\vartheta_1(z,\tau) = -\vartheta_1(-z,\tau). \tag{89}$$

- It transforms under conjugation as follows,

$$\overline{\vartheta_1(z,\tau)} = -\vartheta_1(-\bar{z}, -\bar{\tau}). \tag{90}$$

⋄ It supports the transformation $\tau \to -^1/_\tau$,

$$i\vartheta_1\left(\frac{z}{\tau}, \frac{-1}{\tau}\right) = (-i\tau)^{\frac{1}{2}} e^{\frac{\pi}{\tau} i z^2} \vartheta_1(z,\tau). \tag{91}$$

• It is quasi-periodic,

$$\vartheta_1(z + a + b\tau, \tau) = e^{i\pi(a - b(2z + b\tau + 1))} \vartheta_1(z,\tau) \tag{92}$$

• q-series representation for its logarithm ,

$$\operatorname{Log} \vartheta_1(z,\tau) = \operatorname{Log}(\vartheta_1'(0,\tau)) + \operatorname{Log}(\sin(z)) + 4\sum_{k=1}^{\infty} \frac{e^{i2\pi\tau k}}{k(1 - e^{i2\pi\tau k})} \sin^2(kz) \tag{93}$$

• q-series representation for its logarithm derivative,

$$\frac{\vartheta_1'(z,\tau)}{\vartheta_1(z,\tau)} = \cot(z) + 4\sum_{k=1}^{\infty} \frac{e^{i2\pi\tau k}}{1 - e^{i2\pi\tau k}} \sin(2kz) \tag{94}$$

6.2 Hurwitz and Riemann Zeta functions

For any $q, s \in \mathbb{C}$ with $\operatorname{Re}(q) > 0$ and $\operatorname{Re}(s) > 1$ we let the Hurwitz zeta function be

$$\zeta_q(s) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^s}.$$

This series converges absolutely and for any q it defines an analytic function in s that admits a meromorphic continuation to $\mathbb{C} \setminus \{1\}$. Taking $q = 1$ yields the Riemann zeta function $\zeta(s) = \zeta_1(s)$.

6.3 The Gamma function

The gamma function is defined for all complex numbers except the non-positive integers. For $\operatorname{Re}(z) > 0$, it is defined via the integral:

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x}\, \mathrm{d}x,$$

The recurrence relation $z\Gamma(z) = \Gamma(z+1)$ can be used to extend the integral formulation to all complex numbers z, except the non-positive integers. A useful identity that is related to the Gamma function is:

• For any $A \in \mathbb{N}$ and $z \in \mathbb{C} \setminus (\{0\} \cup \mathbb{Z}^-)$ it holds that

$$\sum_{n=-A}^{A} \operatorname{Log}(z+n) \equiv \operatorname{Log}\left(\frac{\Gamma(A+1+z)\,\Gamma(A+1-z)}{\Gamma(z)\,\Gamma(-z)}\right) - \operatorname{Log} z + \operatorname{Log}(-1)^{A+1} \pmod{2\pi i} \tag{95}$$

Derivation. We first note that due to the recurrence relation of Γ it follows that

$$\sum_{n=0}^{A} \operatorname{Log}(z+n) = \operatorname{Log}\left(\frac{\Gamma(z+A+1)}{\Gamma(z)}\right)$$

It then follows that

$$\begin{aligned}
\sum_{n=-A}^{A} \operatorname{Log}(z+n) &= \sum_{n=0}^{A} (\operatorname{Log}(z+n) + \operatorname{Log}(z-n)) - \operatorname{Log} z \\
&\equiv \sum_{n=0}^{A} (\operatorname{Log}(z+n) + \operatorname{Log}(-z+n) + \operatorname{Log}(-1)) - \operatorname{Log} z \pmod{2\pi i} \\
&\equiv \operatorname{Log}\left(\frac{\Gamma(A+1+z)\,\Gamma(A+1-z)}{\Gamma(z)\,\Gamma(-z)}\right) - \operatorname{Log} z + \operatorname{Log}(-1)^{A+1} \pmod{2\pi i}
\end{aligned}$$

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Appendix D

Calculating the overlap between HFB wave function

1 The HFB wave function

The Hartree-Fock-Bogoliubov (HFB) wave function, $|\Phi\rangle$ is merely a representation of many-quasi-particle state as a vacuum with respect to Bogoliubov quasi-particles operator [1], c_ϵ:

$$c_\epsilon |\Phi\rangle = 0, \quad \text{for all } \epsilon > 0. \tag{1}$$

We note that it's always possible to construct a groundstate that fulfill this condition:

$$|\Phi\rangle = \prod_{\epsilon>0} c_\epsilon |0\rangle \tag{2}$$

[1] Bogoliubov quasi-particles are related to particle operators by a linear transformation that diagonalizes a single-particle Hamiltonian with a particle-hole symmetry (PHS) of a system in consideration

2 Thouless representation of the HFB groundstate

The HFB groundstate can be represented as

$$|\Omega\rangle = A\exp\left(\sum_{i<j} Z_{ij}\psi_i^\dagger\psi_j^\dagger\right)|0\rangle \tag{3}$$

where ψ_i is a fermion annihilation operator satisfying $\psi_i|0\rangle = 0$, Z_{ij} is an skew-symmetric matrix and A is a normalization constant, assuring us that $\langle\Omega|\Omega\rangle = 1$. This representation is known in the literature as the Thouless Representation[1, 2].

Proof:

The similarity transformation that diagonalizes a Hamiltonian that posses a particle-hole symmetry (PHS) is $H_{\text{diagonal}} = P^\dagger HP$ where the unitary matrix P has the form

$$P = \begin{pmatrix} U & \bar{V} \\ V & \bar{U} \end{pmatrix} \tag{4}$$

and the columns of $(U\ V)^T$ are eigenstates that correspond to positive eigenenergies in an acceding order. Thus, the annihilation operator of quasipartices is

$$c_\epsilon = \sum_i (\bar{U}^T_{\epsilon i}\psi_i + \bar{V}^T_{\epsilon i}\psi_i^\dagger) \tag{5}$$

The groundstate must obey $c_\epsilon|\Omega\rangle = 0$ for every $\epsilon > 0$ so we need to find a matrix Z_{ij} the will fulfill this requirement:

$$\begin{aligned} c_\epsilon|\Omega\rangle &\propto \sum_i (\bar{U}^T_{\epsilon i}\psi_i + \bar{V}^T_{\epsilon i}\psi_i^\dagger)\exp\left(\sum_{ij}\frac{1}{2}Z_{ij}\psi_i^\dagger\psi_j^\dagger\right)|0\rangle \\ &= \exp\left(\sum_{ij}\frac{1}{2}Z_{ij}\psi_i^\dagger\psi_j^\dagger\right)\sum_i(\bar{U}^T_{\epsilon i}\psi_i + \bar{V}^T_{\epsilon i}\psi_i^\dagger + \bar{U}^T_{\epsilon k}\sum_j Z_{kj}\psi_j^\dagger)|0\rangle \\ &= \exp\left(\sum_{ij}\frac{1}{2}Z_{ij}\psi_i^\dagger\psi_j^\dagger\right)\sum_i(\bar{V}^T_{\epsilon i} + \sum_j \bar{U}^T_{\epsilon j}Z_{ji})\psi_i^\dagger|0\rangle = 0 \end{aligned} \tag{6}$$

where used the relations $[\psi_k, e^{\sum_{i<j} Z_{ij}\psi_i^\dagger\psi_j^\dagger}] = \partial_{\psi_k^\dagger} e^{\sum_{i<j} Z_{ij}\psi_i^\dagger\psi_j^\dagger}$ and $[\psi_k^\dagger, e^{\sum_{i<j} Z_{ij}\psi_i^\dagger\psi_j^\dagger}] = \partial_{\psi_k} e^{\sum_{i<j} Z_{ij}\psi_i^\dagger\psi_j^\dagger}$ which are valid for a matrix Z_{ij} that is a skew-symmetric[2]. The last equality is true only if

$$\bar{V}^T_{\epsilon i} + \sum_j \bar{U}^T_{\epsilon j}Z_{ji} = 0 \Rightarrow +\bar{V}_{i\epsilon} - \sum_j Z_{ij}\bar{U}_{j\epsilon} = \bar{V} - Z\bar{U} = 0 \Rightarrow Z = (VU^{-1})^*. \tag{7}$$

All that's left is to assure that Z is a skew-symmetric matrix as we assumed. The unitarity of P means that

$$P^\dagger P = 1 \Rightarrow \begin{pmatrix} U^\dagger & V^\dagger \\ V^T & U^T \end{pmatrix}\begin{pmatrix} U & \bar{V} \\ V & \bar{U} \end{pmatrix} = \begin{pmatrix} U^\dagger U + V^\dagger V & U^\dagger\bar{V} + V^\dagger\bar{U} \\ V^T U + U^T V & V^T\bar{V} + U^T\bar{U} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

[2] In order to take the derivative with respect to an operator we must first bring it to the left of the expression by using its commutations relations

multiplying $(P^\dagger P)^*_{1,2}$ by U^{-1} from the right and $(U^{-1})^T$ from the left we find that

$$(U^{-1})^T(U^TV + V^TU)(U^{-1}) = VU^{-1} + (U^{-1})^TV^T = 0 \Rightarrow VU^{-1} = -(VU^{-1})^T. \tag{9}$$

Thus, we verified that $Z = (VU^{-1})^*$ is indeed skew-symmetric.

Three properties that are worth mentioning:

1. The global phase of the groundstate is fixed such that $\langle 0|\Omega_0\rangle = 1$.

2. The matrix Z is gauge-invariant in the sense that all the similarity transformations,$H \mapsto P^\dagger HP$ that diagonalizes the Hamiltonian would result the same matrix Z.

3. In order to exclude one of the operators, $c_{\epsilon'}$ which defines the groundstate through the requirement in Eq.(6), we eliminate the ϵ' column from the matrices V and U. In addition, U is not a square matrix anymore and we regard U^{-1} as a right inverse.

   Proof: Although the columns of U and V are defined up to a global phase, which means that there are infinite number of similarity transformations that would diagonalize H, the matrix Z is unique. We define a general gauge transformation $\mathcal{A}$ which is simply a block diagonal matrix made of arbitrary unitary matrices for degenerate eigenstates and just phase factors for non-degenerate eigenstates. Thus,

$$P \longmapsto P\begin{pmatrix}\mathcal{A} & 0\\ 0 & \bar{\mathcal{A}}\end{pmatrix} \quad \Longrightarrow \quad \begin{matrix} U \longmapsto U\mathcal{A} \\ V \longmapsto V\mathcal{A}\end{matrix} \tag{10}$$

   but Z is unaltered by the gauge transformation,

$$Z = (VU^{-1})^* \longmapsto (V\mathcal{A}(U\mathcal{A})^{-1})^* = (VU^{-1})^*. \tag{11}$$

Derivation of the identities:

$$[\psi_k, e^{\sum\limits_{i<j} Z_{ij}\psi_i^\dagger\psi_j^\dagger}] = \partial_{\psi_k^\dagger} e^{\sum\limits_{i<j} Z_{ij}\psi_i^\dagger\psi_j^\dagger}, \quad [\psi_k^\dagger, e^{\sum\limits_{i<j} Z_{ij}\psi_i^\dagger\psi_j^\dagger}] = \partial_{\psi_k} e^{\sum\limits_{i<j} Z_{ij}\psi_i^\dagger\psi_j^\dagger}. \tag{12}$$

First we recall the Baker-Hausdorff Lemma

$$e^{\lambda B}Ae^{-\lambda B} = A + \lambda[B, A] + \frac{\lambda^2}{2!}[B, [B, A]] + \ldots + \frac{\lambda^n}{n!}[B, [B, [B, \ldots, [B, A]]]] + \ldots.$$

In the case that $[B,[B,A]]=0$ we get $[A,e^{-\lambda B}]=e^{-\lambda B}\lambda[B,A]$. We take $A=\psi_k, B=\sum_{ij}Z_{ij}\psi_i^\dagger\psi_j^\dagger$ and $\lambda=-\frac{1}{2}$ for which,

$$AB=\sum_{ij}Z_{ij}\psi_k\psi_i^\dagger\psi_j^\dagger=\sum_{ij}Z_{ij}\left(\psi_i^\dagger\delta_{ki}-\psi_i^\dagger\left(\delta_{kj}-\psi_j^\dagger\psi_k\right)\right)=\sum_{ij}Z_{ij}\left(\psi_i^\dagger\delta_{ki}-\psi_i^\dagger\delta_{kj}\right)+BA.$$

In addition, $B[B,A]=[B,A]B$ since $\psi_i^\dagger\psi_j^\dagger\psi_k^\dagger=\psi_k^\dagger\psi_i^\dagger\psi_j^\dagger$ so $[B,[B,A]]=0$. Now assuming that Z is a skew-symmetric matrix we get

$$\left[\psi_k,e^{\frac{1}{2}Z_{ij}\psi_i^\dagger\psi_j^\dagger}\right]=\frac{1}{2}e^{\frac{1}{2}B}\sum_i\left(Z_{ki}-Z_{ik}\right)\psi_i^\dagger=e^{\frac{1}{2}B}\sum_i Z_{ki}\psi_i^\dagger=\frac{\partial}{\partial\psi_k^\dagger}e^{\frac{1}{2}B}\tag{13}$$

where in order take the derivative with respect to an operator we must first anticommute it through until it adjacent to the derivative.

Next, we take $A=\psi_k^\dagger, B=\sum_{ij}Z_{ij}\psi_i^\dagger\psi_j^\dagger$ and $\lambda=-\frac{1}{2}$. Since $\psi_i^\dagger\psi_j^\dagger\psi_k^\dagger=\psi_k^\dagger\psi_i^\dagger\psi_j^\dagger$, $BA=AB$ and we get $\left[\psi_k^\dagger,e^{\frac{1}{2}\sum_{ij}Z_{ij}\psi_i^\dagger\psi_j^\dagger}\right]=\frac{\partial}{\partial\psi_k}e^{\frac{1}{2}B}=0$.

3 Thouless representation of many-quasi-particle states

The trick is to represent quasi-particle excitations as HFB vacua by exchanging the role of particle and hole creation operators [1, 3, 4]. Let us elucidate this statement by constructing an one-quasi-particle state. We take, for instance, with a fully paired vacuum

$$|\Phi_0\rangle=\prod_{i=1}^{N}c_{\epsilon_i}|0\rangle\tag{14}$$

and add a single excitation

$$|\Phi_1\rangle=c_{\epsilon_1}^\dagger|\Phi_0\rangle\tag{15}$$

This one-quasi-particle state is a vacuum to the operators $(\tilde{c}_{\epsilon_1},\tilde{c}_{\epsilon_2},...,\tilde{c}_{\epsilon_N})$ with

$$\tilde{c}_1=c_{\epsilon_1}^\dagger,\tilde{c}_{\epsilon_2}=c_{\epsilon_2},...,\tilde{c}_{\epsilon_N}=c_{\epsilon_N}.\tag{16}$$

This is merely an exchange of a quasi-particle particle annihilation operator, c_{ϵ_1} with its hole counterpart, $c_{-\epsilon_1}$. This can easily understood by: a) writing explicitly the basis transformation of the creation-annihilation operators $c_\epsilon^\dagger=\sum_i\left(u_\epsilon^i\psi_i^\dagger+v_\epsilon^i\psi_i\right)$ and $c_\epsilon=\sum_i\left(\bar{u}_\epsilon^i\psi_i+\bar{v}_\epsilon^i\psi_i^\dagger\right)$. b) noticing that the PHS implies that $c_{-\epsilon}^\dagger=\sum_i\left(\bar{v}_\epsilon^i\psi_i^\dagger+\bar{u}_\epsilon^i\psi_i\right)$ which leads to a relationship between the creation-annihilation operators, $c_\epsilon^\dagger=c_{-\epsilon}$. Thus, the exchange is practically obtained by replacing columns 1 in the matrices U and V by columns 1 in the matrices $\bar{V}$ and $\bar{U}$, respectively. In order to obtain a n quasi-particles state from a fully paired groundstate, we repeat the transformation,

$$(U_{ij}, V_{ij}) \longleftrightarrow (\bar{V}_{ij}, \bar{U}_{ij}) \text{ with } 1 \le i \le N, \tag{17}$$

for n different columns. In the special case of a system with zero energy single-particle excitations, this method can be used to change the parity of the groundstate.

4 The evolution of a many-quasi-particle states

Here we describe a procedure to evolve in time many-quasi-particle state in terms of single particle evolution. Since quasi-particle excitations can be represented as HFB vacua, $|\Phi\rangle$ with respect to a set of Bogoliubov quasi-particles operator that fulfill the condition

$$c_\epsilon|\Phi\rangle = 0, \quad \text{for all } \epsilon > 0, \tag{18}$$

the evolution of the wave function is determined by the evolution of each Bogoliubov quasi-particle *and the vacuum, if it's not an eigenstate of the Hamiltonian.*[3] The quasi-particle operators at time t are

$$\mathbf{c}_\epsilon(t) = P^\dagger(0)\mathcal{U}^\dagger(t)\psi_x(0) \equiv P^\dagger(t)\psi_x(0) \tag{19}$$

where $\psi_x(0) = (\psi_{x_1}, \psi_{x_2} \dots \psi_{x_N}, \psi^\dagger_{x_1}, \psi^\dagger_{x_2} \dots \psi^\dagger_{x_N})^T$ and $\mathbf{c}_\epsilon(t) = (c_{\epsilon_1,t}, c_{\epsilon_2,t} \dots c_{\epsilon_N,t}, c^\dagger_{-\epsilon_1,t}, c^\dagger_{-\epsilon_2,t} \dots c^\dagger_{-\epsilon_N,t})^T$ are Nambu spinors, $\mathcal{U}(t) = e^{-iHt}$ is the evolution operator and $P(0)$ is the diagonalizing matrix of the Hamiltonian at time $t = 0$ which the has form $P(0) = \begin{pmatrix} U(0) & \bar{V}(0) \\ V(0) & \bar{U}(0) \end{pmatrix}$. Thus, the wave function $|\Phi\rangle$ at time t is

$$|\Phi(t)\rangle = A \exp\left(\sum_{i<j} Z_{ij}\psi^\dagger_i\psi^\dagger_j\right)|0(t)\rangle, \tag{20}$$

where $Z(t) = (V(t)U^{-1}(t))^*$ and A is a normalization constant.

One last caveat - a groundstate at time $t > 0$ is possibly an excited state with respect to the quasi-particle operators at time $t = 0$.

5 Thousless' theorem for changing a reference vacuum

Starting with a general product wave functions $|\Phi_0\rangle$ which is the vacuum to quasi-particle operators β that posses PHS ($\beta^\dagger_{-\epsilon} = \beta_\epsilon$), any other general product wave function $|\Phi_1\rangle$ which is not orthogonal to $|\Phi_0\rangle$ may be expressed in

[3] If the vacuum state also evolves in time, as it is not an eigenstate of the Hamiltonian, the procedure described here is not straightforward to implement and, probably, use of many-body evolution operator is unavoidable.

the form

$$|\Phi_1\rangle = A\exp\left(\sum_{i<j} Z_{ij}\beta_i^\dagger\beta_j^\dagger\right)|\Phi_0\rangle \tag{21}$$

where Z is a skew-symmetric matrix and A is a normalization constant.[1, 2]

Proof:

To prove this theorem we start with two sets of quasi-particle operators $\beta, \beta^\dagger$ and $\gamma, \gamma^\dagger$ and their groundstates, $|\Phi_0\rangle$ and $|\Phi_1\rangle$,respectively. The two sets are related to a common set of fermion operators $c, c^\dagger$ by two unitary transformations $P_{(0)}$ and $P_{(1)}$ that posses a PHS[4]:

$$\boldsymbol{\beta}^\dagger = \mathbf{c}^\dagger P_{(0)} \text{ and } \boldsymbol{\gamma}^\dagger = \mathbf{c}^\dagger P_{(1)}, \tag{22}$$

where $P_{(i)} = \begin{pmatrix} U_{(i)} & \bar{V}_{(i)} \\ V_{(i)} & \bar{U}_{(i)} \end{pmatrix}$ with $i = 1, 2$ and $\boldsymbol{c}^\dagger = (c^\dagger\ c)$, $\boldsymbol{\beta}^\dagger = (\beta^\dagger\ \beta)$ and $\boldsymbol{\gamma}^\dagger = (\gamma^\dagger\ \gamma)$ are Nambu spinor, i.e. $\boldsymbol{c} = \tau_x K \boldsymbol{c}^\dagger$ (K is the complex conjugation operator). Next, we express the operators $\gamma, \gamma^\dagger$ in terms of the operators $\beta, \beta^\dagger$:

$$\gamma^\dagger = \beta^\dagger P_{(0)}^\dagger P_{(1)} \equiv \beta^\dagger P_{(0,1)} \tag{23}$$

where P in terms of the block matrices $U_{(i)}$ and $V_{(i)}$ is

$$P_{(0,1)} = P_{(0)}^\dagger P_{(1)} = \begin{pmatrix} U_{(0)}^\dagger U_{(1)} + V_{(0)}^\dagger V_{(1)} & U_{(0)}^\dagger \bar{V}_{(1)} + V_{(0)}^\dagger \bar{U}_{(1)} \\ V_{(0)}^T U_{(1)} + U_{(0)}^T V_{(1)} & V_0^T \bar{V}_{(1)} + U_{(0)}^T \bar{U}_{(1)} \end{pmatrix} \equiv \begin{pmatrix} U_{(0,1)} & \bar{V}_{(0,1)} \\ V_{(0,1)} & \bar{U}_{(0,1)} \end{pmatrix} \tag{24}$$

The Onishi formula shows that the norm of the overlap between the two states $|\Phi_0\rangle$ and $|\Phi_1\rangle$ is $\sqrt{|\det U_{(0,1)}|}$. Thus, non orthogonality of $|\Phi_0\rangle$ and $|\Phi_1\rangle$ means that we can invert $U_{(0,1)}$[5]. We proceed by defining a transformation that preserves the PHS and does not mix the creation and annihilation operators $\gamma_\epsilon^\dagger, \gamma_\epsilon$ ($\gamma_\epsilon = \gamma_{-\epsilon}^\dagger$):

$$\begin{pmatrix} \tilde{\gamma}_\epsilon^\dagger & \tilde{\gamma}_{-\epsilon}^\dagger \end{pmatrix} = \begin{pmatrix} \gamma_\epsilon^\dagger & \gamma_{-\epsilon}^\dagger \end{pmatrix} \begin{pmatrix} U_{(0,1)}^{-1} & 0 \\ 0 & \bar{U}_{(0,1)}^{-1} \end{pmatrix} = \begin{pmatrix} \beta_\epsilon^\dagger & \beta_{-\epsilon}^\dagger \end{pmatrix} \begin{pmatrix} I & Z_{(0,1)} \\ \bar{Z}_{(0,1)} & I \end{pmatrix} \tag{25}$$

where $Z_{(0,1)} \equiv \left(V_{(0,1)} U_{(0,1)}^{-1}\right)^*$ and in the last equality we simply used the relation between $\boldsymbol{\gamma}^\dagger$ and $\boldsymbol{\beta}^\dagger$. Since γ and $\tilde{\gamma}$ share the same vacuum, $|\Phi_1\rangle$, it's enough to show that $\tilde{\gamma}$ annihilates the Thouless' represenion of $|\Phi_1\rangle$ with respect to the reference wave function $|\Phi_0\rangle$, appearing in Eq.(21):

$$\begin{aligned} \tilde{\gamma}_\epsilon \exp\left(\frac{1}{2} Z_{ij}\beta_i^\dagger\beta_j^\dagger\right)|\Phi_0\rangle &= (Z_{i\epsilon}\beta_i^\dagger + I_{i\epsilon}\beta_i)\exp\left(\frac{1}{2} Z_{ij}\beta_i^\dagger\beta_j^\dagger\right)|\Phi_0\rangle \\ &= \exp\left(\frac{1}{2} Z_{ij}\beta_i^\dagger\beta_j^\dagger\right)(Z_{i\epsilon}\beta_i^\dagger + \beta_\epsilon + \frac{1}{2} Z_{\epsilon j}\beta_j^\dagger - \frac{1}{2} Z_{i\epsilon}\beta_i^\dagger)|\Phi_0\rangle = 0 \end{aligned} \tag{26}$$

[4] We assume $\dim\left(P^{(0)}\right) = \dim\left(P^{(1)}\right)$.

[5] We recall that $U_{(0,1)}$ is invertible if only if $\det U_{(0,1)\neq 0}$

where we used the Einstein summation rule and Z refers to $Z_{(0,1)}$. As explained in section 3, every many-quasi-particle state can be represented as HFB vacua for properly defined new quasi-particle operators. This trick, which completes the proof, is valid for any chosen reference frame by applying it on the $\gamma^\dagger_\epsilon, \gamma_\epsilon$ operators.

A few remarks that are worth mentioning:

1. The normalization constant is given by the Onishi formula (explained in the following sections),

$$A = \sqrt{\langle\phi_1|\phi_1\rangle} = \sqrt{|\det U_{(0,1)}|} = |\langle\Phi_0|\Phi_1\rangle|, \tag{27}$$

   where $|\phi_i\rangle$ is the unnormalized state that correspond to $|\Phi_i\rangle$.

2. The global phase of HFB states in the Thouless representation has implicitly fixed by requiring $\langle\Phi_0|\Phi_i\rangle = 1$, where $|\Phi_0\rangle$ is the common reference frame of the states $|\Phi_i\rangle$. This means that the phase of the states $|\Phi_i\rangle$ is always relative to the phase of $|\Phi_0\rangle$, but the phase of the overlap between two different state, $\langle\Phi_i|\Phi_j\rangle$ do not depend on their common reference state.

3. When $\langle\Phi_0|\Phi_i\rangle = 0$ the best practical strategy is to use another reference wave function instead. If the aim is to calculate the overlap $\langle\Phi_i|\Phi_j\rangle$ than the new reference HFB wave function has to be close to both $|\Phi_i\rangle$ and $|\Phi_j\rangle$.

4. The matrix Z itself is not gauge-invariant since it depends on the global phases of the single-particle eigenstates of the reference system. However, the overlap between two many-quasi-particle states that share a common reference wave function is gauge-invariant. Thus, the overlap does not depend on the global phases that multiply the columns of $P_{(0)}$ and $P_{(1)}$.
   Proof: We define a general gauge transformation $\mathcal{A}$ which is simply a block diagonal matrix made of arbitrary unitary matrices for degenerate eigenstates and just phase factors for non-degenerate eigenstates,

$$P_{(i)} \longmapsto P_{(i)}\begin{pmatrix} \mathcal{A}_{(i)} & 0 \\ 0 & \bar{\mathcal{A}}_{(i)} \end{pmatrix} \quad \Longrightarrow \quad \begin{matrix} U \longmapsto U\mathcal{A}_{(i)} \\ V \longmapsto V\mathcal{A}_{(i)} \end{matrix}. \tag{28}$$

   Thus,

$$\begin{aligned} P_{(0,1)} = P^\dagger_{(0)}P_{(1)} \quad &\longmapsto \quad P'_{(0,1)} = \begin{pmatrix} \bar{\mathcal{A}}_{(0)} & 0 \\ 0 & \mathcal{A}_{(0)} \end{pmatrix} P^\dagger_{(0)}P_{(1)} \begin{pmatrix} \mathcal{A}_{(1)} & 0 \\ 0 & \bar{\mathcal{A}}_{(1)} \end{pmatrix} \\ &\Longrightarrow \quad \begin{matrix} U'_{(01)} = \bar{\mathcal{A}}_{(0)}U_{(01)}\mathcal{A}_{(1)} \\ V'_{(01)} = \mathcal{A}_{(0)}V_{(01)}\mathcal{A}_{(1)} \end{matrix} \\ &\Longrightarrow \quad \bar{Z}'_{(01)} = V'_{(01)}U'^{-1}_{(01)} = \mathcal{A}_0 V_{(01)}\mathcal{A}_1\bar{\mathcal{A}}_1 U^{-1}_{(01)}\mathcal{A}_0 = \mathcal{A}_0 V_{(01)}U^{-1}_{(01)}\mathcal{A}_0. \end{aligned} \tag{29}$$

By Robledo's formula, The overlap of the two unnormalized states is proportional to

$$\begin{aligned}
\mathrm{Pf}\begin{pmatrix} Z'_{(01)} & -I \\ I & -\bar{Z}'_{(02)} \end{pmatrix} &= \mathrm{Pf}\left[\begin{pmatrix} 0 & \bar{\mathcal{A}}_0 \\ \mathcal{A}_0 & 0 \end{pmatrix}\begin{pmatrix} Z_{(01)} & -I \\ I & -\bar{Z}_{(02)} \end{pmatrix}\begin{pmatrix} 0 & \mathcal{A}_0 \\ \bar{\mathcal{A}}_0 & 0 \end{pmatrix}\right] && (30)\\
&= \mathrm{Pf}\begin{pmatrix} Z_{(01)} & -I \\ I & -\bar{Z}_{(02)} \end{pmatrix}\det\begin{pmatrix} 0 & \mathcal{A}_0 \\ \bar{\mathcal{A}}_0 & 0 \end{pmatrix} && (31)\\
&= \mathrm{Pf}\begin{pmatrix} Z_{(01)} & -I \\ I & -\bar{Z}_{(02)} \end{pmatrix}. && (32)
\end{aligned}$$

In the last step we used the following properties:

(a) $\det(cA) = c^n \det(A)$ where $n = \dim(A)$.

(b) $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)$ when $CD = DC$.

(c) Simultaneous interchange of two different rows and corresponding columns changes the sign of the Pfaffian.

In addition, we give another proof by calculating the overlap of two unnormalized states using Read's formula (which is more familiar but accurate only up to a sign),

$$\sqrt{\det(I - \bar{Z}'_{(01)}Z'_{(02)})} = \sqrt{\det(\mathcal{A}_{(0)}(I - V_{(01)}U^{-1}_{(01)}\bar{V}_{(02)}\bar{U}^{-1}_{(02)})\bar{\mathcal{A}}_{(0)})} = \sqrt{\det(I - \bar{Z}_{(01)}Z_{(02)})}. \tag{33}$$

6 Robledo's formula for the overlap of HFB wave functions

The overlap between two unnormalized HFB wave functions,

$$|\phi_k\rangle = \exp\left(\sum_{i<j} Z^{(k)}_{ij}\beta^\dagger_i\beta^\dagger_j\right)|\Phi_0\rangle \quad \textit{with } k = 1, 2 \tag{34}$$

that share a common set of quasi-particle operators and a corresponding reference wave function is

$$\langle\phi_1|\phi_2\rangle = S_N pf\,\mathcal{Z} \tag{35}$$

it terms of the phase $S_N = (-1)^{N(N+1)/2}$ *and the* $2N \times 2N$ *skew-symmetric matrix*

$$\mathcal{Z} = \begin{pmatrix} Z^{(2)} & -\mathbb{I} \\ \mathbb{I} & -\bar{Z}^{(1)} \end{pmatrix}. \tag{36}$$

Proof:

The proof follows Robledo's mathematically elegant formulation which is based on fermion coherent states[5].

Intermezzo - fermion coherent states. The fermion coherent states $|\zeta\rangle$ are defined as eigenstates of the quasi-particle annihilation operators,

$$\beta_k|\zeta\rangle = \zeta_k|\zeta\rangle \tag{37}$$

and similarly, the adjoint of the coherent state is

$$\langle\zeta|\beta_k^\dagger = \zeta_k^*\langle\zeta|, \tag{38}$$

where the eigenvalues ζ_k and ζ_k^* are generators in Grassmann algebra that are not connected by complex conjugation. The coherent states satisfy a closure relation

$$\mathbb{I} = \int \mathrm{d}\mu(\zeta)|\zeta\rangle\langle\zeta| \tag{39}$$

where the metric of the integral is given by $\mathrm{d}\mu(\zeta) = e^{-\sum_i \zeta_i^*\zeta_i}\prod_j \mathrm{d}\zeta_j^*\mathrm{d}\zeta_j$. The subject of fermion coherent states is covered in many textbooks about many-body quantum systems[6, 7, 8].

Evaluation of the overlap. To calculate the overlap between two unnormalized HFB wave functions,

$$|\phi_k\rangle = \exp\left(\sum_{i<j} Z_{ij}^{(k)}\beta_i^\dagger\beta_j^\dagger\right)|\Phi_0\rangle \quad \text{with } k = 1, 2 \tag{40}$$

that share a common set of quasi-particle operators and a corresponding reference wave function, the closure relation Eq.(39) is inserted to obtain

$$\langle\phi_1|\phi_2\rangle = \int \mathrm{d}\mu(\zeta)\langle\Phi_0|\exp\left(\frac{1}{2}\sum_{ij}\bar{Z}_{ij}^{(1)}\beta_j\beta_i\right)|\zeta\rangle\langle\zeta|\exp\left(\frac{1}{2}\sum_{ij}Z_{ij}^{(2)}\beta_i^\dagger\beta_j^\dagger\right)|\Phi_0\rangle \tag{41}$$

Using now Eqs. (37) and (38) one arrives to

$$\langle\phi_1|\phi_2\rangle = \int \mathrm{d}\mu(\zeta)\exp\left(\frac{1}{2}\sum_{ij}\bar{Z}_{ij}^{(1)}\zeta_j\zeta_i\right)\exp\left(\frac{1}{2}\sum_{ij}Z_{ij}^{(2)}\zeta_i^*\zeta_j^*\right) \tag{42}$$

where the property $|\langle\Phi_0|\zeta\rangle|^2 = 1$ is used[6]. The above integral can can be written in a more compact way by introducing the Nambu spinor $\zeta^T = (\zeta_1^*, \zeta_2^*, ...\zeta_N^*, \zeta_1, \zeta_2, ...\zeta_N)$ and a skew symmetric matrix,

$$\mathcal{Z} = \begin{pmatrix} Z^{(2)} & -I \\ I & -\bar{Z}^{(1)} \end{pmatrix} \tag{43}$$

as

$$\langle\phi_1|\phi_2\rangle = \int \prod_i(\mathrm{d}\zeta_i^*\mathrm{d}\zeta_i)\exp\left(\frac{1}{2}\zeta^T\mathcal{Z}\zeta\right). \tag{44}$$

[6]This is easy to deduce from the explicit form of fermion coherent state, $\prod_i(1-\zeta_i\beta_i^\dagger|\Phi_0\rangle$ and recalling that the operators and generators fulfill the relations: $\{\zeta_i, c_j\} = 0$ and $(\zeta_i c_j)^\dagger = c_j^\dagger\zeta_i^*$ where ζ_i denotes a Grassmann variable and c_j is an operator.

The skew-symmetric matrix $\mathcal{Z}$ can always be transformed to canonical form by means of a unitary transformation M,

$$\mathcal{Z} = M \begin{pmatrix} 0 & \dots & 0 & \lambda_1 & 0 & 0 \\ \vdots & \ddots & \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & 0 & 0 & \lambda_N \\ -\lambda_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & -\lambda_N & 0 & \dots & 0 \end{pmatrix} M^T \equiv M \mathcal{Z}_c M^T \tag{45}$$

where $\lambda_1, \dots, \lambda_N$ are non-negative real numbers[9]. Moreover, the new Nambu spinor $\eta^T = (\eta_1^*, \dots, \eta_N^*, \eta_1, \dots, \eta_N) = \zeta^T M$ retains the structure of the original spinor ζ. The overlap in Eq. (44) becomes

$$\langle \phi_1 | \phi_2 \rangle = \det M \int \prod_i (\mathrm{d}\eta_i^* \mathrm{d}\eta_i) \exp\left(\frac{1}{2}\eta^T \mathcal{Z}_c \eta\right) = \\ \det M \int \prod_i (\mathrm{d}\eta_i^* \mathrm{d}\eta_i) \exp\left(\sum_{i=1}^N \lambda_i \eta_i^* \eta_i\right) = (-1)^N \det M \prod_{i=1}^N \lambda_i, \tag{46}$$

where the Jacobian that correspond to the transformation is $\det^{-1}(\bar{M}) = \det(M)$, as opposed to the Jacobian of complex numbers algebra which would be $\det(\bar{M})$. Moreover, the factor $(-1)^N$ originates from the integration over a Grassmannian Gaussian, $\int \mathrm{d}\eta_i^* \mathrm{d}\eta_i e^{\lambda_i \eta_i^* \eta_i} = -\lambda_i$. The final expression can be cast in terms of the pfaffian of a skew-symmetric matrix. The connection between the product of λ_i's and the pfaffian reads

$$\prod_{i=1}^N \lambda_i = (-1)^{N(N-1)/2} \mathrm{pf}(\mathcal{Z}_c) \tag{47}$$

and is a consequence of the relation[7]

$$\det(R) = (-1)^{N(N-1)/2} \mathrm{pf}\begin{pmatrix} 0 & R \\ -R^T & 0 \end{pmatrix}, \tag{48}$$

and we take the matrix R to be

$$R = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}. \tag{49}$$

Using the property

$$\det(M) \mathrm{pf}(\mathcal{Z}_c) = \mathrm{pf}(M^T \mathcal{Z}_c M) = \mathrm{pf}(\mathcal{Z}) \tag{50}$$

[7] $\mathcal{Z}_c = i\sigma_y \otimes A$ where A is a diagonal matrix with entries $\lambda_1, \dots, \lambda_N$. However, $\mathcal{Z}$ can also brought to another canonical forms, $\tilde{\mathcal{Z}}_c = A \otimes i\sigma_y$. In this case $\prod_{i=1}^N \lambda_i = \mathrm{pf}(\tilde{\mathcal{Z}}_c)$ and the factor $(-1)^{N(N-1)/2}$ would raise from the different integration order, $(\prod_{k=1}^N \eta_k^*)(\prod_{k=1}^N \eta_k) = (-1)^{N(N-1)/2} \prod_{k=1}^N (\eta_k^* \eta_k)$. Explanation of the law of linear transformation can be found in page 35 in of Reference [7].

the final result is obtained,

$$\langle\phi_1|\phi_2\rangle = s_N \mathrm{pf}\begin{pmatrix} Z^{(2)} & -I \\ I & -\bar{Z}^{(1)} \end{pmatrix}, \tag{51}$$

where $s_N = (-1)^{N(N+1)/2}$.

7 The overlap norm of HFB states (Onishi formula)

Using the relation

$$\mathrm{pf}(A)^2 = \det(A), \tag{52}$$

to write Robledo's formula in terms of a determinant yields

$$\langle\phi_1|\phi_2\rangle = s_N \mathrm{pf}(\mathcal{Z}) \propto \sqrt{\det\begin{pmatrix} Z^{(2)} & -I \\ I & -\bar{Z}^{(1)} \end{pmatrix}}. \tag{53}$$

The right expression in the equality gives overlap of the two states only up to a sign as consequence of the determinant being equal to the square of a pfaffian. Our next step is to reduce the dimension of matrix $\mathcal{Z}$ for which we calculate the determinant. This is achieved by using an identity for block matrices,

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC) \tag{54}$$

where A,B,C and D are square matrices of the same order with complex coefficients and $CD = DC$. Thus, up to a sign, the overlap between two states reads

$$\langle\phi_1|\phi_2\rangle \propto \det(I - Z^{(2)}\bar{Z}^{(1)}) = \det(I - \bar{Z}^{(1)}Z^{(2)}), \tag{55}$$

where in the last equality we used Sylvester's determinant identity[10],

$$\det(I + AB) = \det(I + BA). \tag{56}$$

Eq. (55) is known as Onishi formula which is the usual expression for the norm.

Finally we are in position to write a simple expression for the normalization constant of a HFB wave function,

$$\begin{aligned} \langle\phi_1|\phi_1\rangle &= \sqrt{\det\left(I - \bar{Z}^{(1)}Z^{(1)}\right)} = \sqrt{\det\left(I + (U^T_{(1)})^{-1}V^T_{(1)}\bar{V}_{(1)}\bar{U}^{-1}_{(1)}\right)} \\ &= \sqrt{\det\left(U^T_{(1)})^{-1}\left(U^T_{(1)}\bar{U}_{(1)} + V^T_{(1)}\bar{V}_{(1)}\right)\bar{U}^{-1}_{(1)}\right)} = \sqrt{\det\left((U^T_{(1)})^{-1}\bar{U}^{-1}_{(1)}\right)} \\ &= \sqrt{\det\left((U^{-1}_{(1)})^T\bar{U}^{-1}_{(1)}\right)} = \left|\det U^{-1}_{(1)}\right| = \left|\frac{1}{\det U_{(1)}}\right| \end{aligned} \tag{57}$$

where $U_{(0)}^T \bar{U}_{(0)} + V_{(0)}^T \bar{V}_{(0)} = I$ because $P_{(0)}^\dagger P_{(0)} = I$ (see Eq.(8)).

Similarly, up to a sign, the overlap between two different unnormalized HFB states is,

$$\begin{aligned}
\langle \phi_1 | \phi_2 \rangle &= \sqrt{\det\left(I - \bar{Z}^{(1)} Z^{(2)}\right)} = \sqrt{\det\left(I + (U_{(1)}^T)^{-1} V_{(1)}^T \bar{V}_{(2)} \bar{U}_{(2)}^{-1}\right)} \\
&= \sqrt{\det\left((U_{(1)}^T)^{-1}\left(U_{(1)}^T \bar{U}_{(2)} + V_{(1)}^T \bar{V}_{(2)}\right) \bar{U}_{(2)}^{-1}\right)} = \sqrt{\det\left((U_{(1)}^T)^{-1}\, U_{(2,1)}\, \bar{U}_{(2)}^{-1}\right)} \\
&= \sqrt{\det\left((U_{(1)}^{-1})^T U_{(2,1)} \bar{U}_{(2)}^{-1}\right)} = \sqrt{\frac{\det U_{(2,1)}}{\det \bar{U}_{(2)} \det U_{(1)}}}.
\end{aligned} \tag{58}$$

By using Eq.(57), we obtain the overlap between two different normalized HFB states -

$$\langle \Phi_1 | \Phi_2 \rangle = \sqrt{\frac{\left|\det U_{(2)}\right| \left|\det U_{(1)}\right|}{\det \bar{U}_{(2)} \det U_{(1)}}} \sqrt{\det U_{(2,1)}}. \tag{59}$$

The first root can contribute only a phase as its magnitude is always one. The second root is invariant under the change of a reference wave function as $P_{(2,1)} \equiv P_{(2)}^\dagger P_{(1)} = P_{(0,2)}^\dagger P_{(0,1)}$.

The analog of Onishi formula for a regular metal

The many-body ground state of the system is

$$|\Omega(t)\rangle = \prod_{\epsilon \le \mu} c_\epsilon^\dagger(t)|0\rangle = \prod_{\epsilon \le \mu} \sum_j \bar{V}_\epsilon^j \psi_j^\dagger(t)|0\rangle = \det\left(V'^\dagger(t)\right) \prod_j \psi_j^\dagger |0\rangle. \tag{60}$$

where $V'_{i,j} = \begin{cases} V_{i,j}, & j \le \mu \\ 0 & j > \mu \end{cases}.$

Using the result above we find that the overlap of $\langle \Omega(0) | \Omega(t) \rangle$ is

$$\langle \Omega(0) | \Omega(t) \rangle = \langle \Omega(0) | \left(\prod_j \psi_j^\dagger\right)^\dagger \det\left(V'(0)\right) \det\left(V'^\dagger(t)\right) \prod_j \psi_j^\dagger |0\rangle = \det\left(V'^\dagger(t) V'(0)\right), \tag{61}$$

which is the analog of Onishi formula for a regular metal.

Proving the identity - $P_{(2,1)} \equiv P_{(2)}^\dagger P_{(1)} = P_{(0,2)}^\dagger P_{(0,1)}$.

In order to prove the relation it is enough to show that:

1. $U_{(1,2)} = U_{(0,1)}^\dagger U_{(0,2)} + V_{(0,1)}^\dagger V_{(0,2)}$

Proof:

$$(U_0^\dagger U_1 + V_0^\dagger V_1)^\dagger (U_0^\dagger U_2 + V_0^\dagger V_2) + (V_0^T U_1 + U_0^T V_1)^\dagger (V_0^T U_2 + U_0^T V_2) =$$

$$(U_1^\dagger U_0 + V_1^\dagger V_0)(U_0^\dagger U_2 + V_0^\dagger V_2) + (U_1^\dagger \bar{V}_0 + V_1^\dagger \bar{U}_0)(V_0^T U_2 + U_0^T V_2) =$$

$$U_1^\dagger \underbrace{(U_0 U_0^\dagger + \bar{V}_0 V_0^T)}_{I} U_2 + V_1^\dagger \underbrace{(V_0 V_0^\dagger + \bar{U}_0 U_0^T)}_{I} V_2 +$$

$$U_1^\dagger \underbrace{(U_0 V_0^\dagger + \bar{V}_0 U_0^T)}_{0} V_2 + V_1^\dagger \underbrace{(V_0 U_0^\dagger + \bar{U}_0 V_0^T)}_{0} U_2 =$$

$$U_1^\dagger U_2 + V_1^\dagger V_2 = U_{1,2} \tag{62}$$

2. $V_{(1,2)} = V_{(0,1)}^T U_{(0,2)} + U_{(0,1)}^T V_{(0,2)}$

Proof:

$$(V_0^T U_1 + U_0^T V_1)^T (U_0^\dagger U_2 + V_0^\dagger V_2) + (U_0^\dagger U_1 + V_0^\dagger V_1)^T (V_0^T U_2 + U_0^T V_2) =$$

$$(U_1^T V_0 + V_1^T U_0)(U_0^\dagger U_2 + V_0^\dagger V_2) + (U_1^T \bar{U}_0 + V_1^T \bar{V}_0)(V_0^T U_2 + U_0^T V_2) =$$

$$U_1^T \underbrace{(V_0 U_0^\dagger + \bar{U}_0 V_0^T)}_{0} U_2 + U_1^T \underbrace{(V_0 V_0^\dagger + \bar{U}_0 U_0^T)}_{I} V_2 +$$

$$V_1^T \underbrace{(U_0 U_0^\dagger + \bar{V}_0 V_0^T)}_{I} U_2 + V_1^T \underbrace{(U_0 V_0^\dagger + \bar{V}_0 U_0^T)}_{0} V_2 =$$

$$U_1^T V_2 + V_1^T U_2 = V_{1,2} \tag{63}$$

8 Zero-modes

The standard procedure for calculating the overlap between two many-body states assume that all positive energy single-particle eigenstates are related to the negative ones by the PHS operator. However, it is not granted that degenerate eigenstates would obey this relation and one must construct such states. The case of zero-modes is more complicated, since not only that they are degenerate, it is not obvious which one of the zero-modes constructs the many-body groundstate. In this section we describe a procedure to construct zero-modes which are related by the PHS operator and a scheme to determine the zero-mode that build to many-body groundstate.

8.1 Construction of zero-modes related by the PHS operator

The PHS, $H = \tau_x H^* \tau_x$, assures us that two non-degenerate states with energy $|\epsilon| > 0$ are related by $|\epsilon\rangle = \tau_x K|-\epsilon\rangle$. In the case of zero-modes, $\epsilon = 0$ we need to *construct two orthonormal states that also maintain this relation*,

$$|v_1\rangle = a|u_1\rangle + b|u_2\rangle \tag{64}$$

$$|v_2\rangle = Tv_1 = \bar{a}T|u_1\rangle + \bar{b}T|u_2\rangle \tag{65}$$

where T is the PHS operator, $T = \tau_x K$ (K is complex conjugate operator).
The requirement that v_1 (and v_2) are normalized gives a constrain on a and b,

$$\langle v_1|v_1\rangle = 1 \Rightarrow |a|^2 + |b|^2 = 1 \Rightarrow a = |\cos\alpha|e^{i\beta}, \quad b = |\sin\alpha|e^{i\gamma}. \tag{66}$$

The second requirement, namely, that the two zero-mode are orthogonal yields the following constrain:

$$\langle v_1|v_2\rangle = \bar{a}^2\langle u_1|T|u_1\rangle + \bar{b}^2\langle u_2|T|u_2\rangle + \bar{a}\bar{b}\left(\langle u_1|T|u_2\rangle + \langle u_2|T|u_1\rangle\right) = 0. \tag{67}$$

Using the identity,

$$\langle u_2|T|u_1\rangle = \langle Tu_1|u_2\rangle^* = \langle u_1|T^\dagger|u_2\rangle = \langle u_1|T|u_2\rangle \tag{68}$$

the relation can be further simplified,

$$\langle v_1|v_2\rangle = \bar{a}^2\underbrace{\langle u_1|T|u_1\rangle}_{A} + \bar{b}^2\underbrace{\langle u_2|T|u_2\rangle}_{B} + \bar{a}\bar{b}\underbrace{2\langle u_1|T|u_2\rangle}_{C} = 0. \tag{69}$$

In the case of $C = 0$, the constrain is simplified further to $A\bar{a}^2 + B\bar{b}^2 = 0$ and together with the first constrain $|a|^2 + |b|^2 = 1$ we get

$$|a|^2 e^{-2\beta}A + (1 - |a|^2)e^{-i2\gamma}B = 0 \Rightarrow |a|^2(Ae^{-i2\beta} - Be^{-i2\gamma}) + Be^{-i2\gamma} = 0 \Rightarrow |a|^2 = \frac{1}{1 - \frac{A}{B}e^{i2(\gamma-\beta)}} \tag{70}$$

with $\gamma - \beta = -\frac{1}{2}\arg\frac{A}{B} + (n + \frac{1}{2})\pi$ and $n \in \mathbb{Z}$ because $|a| < 1$. Since the zero-modes are defined up to a phase it's enough to determine the relative phase between a and b.[8]

If $C \neq 0$, combing the two constrains (Eq.66 and Eq.69) yields,

$$\underbrace{|a|^2}_{x}\underbrace{e^{-i2\beta}A}_{A'} + (1 - |a|^2)\underbrace{e^{-i2\gamma}B}_{B'} + |a|\sqrt{1 - |a|^2}\underbrace{Ce^{-i(\beta+\gamma)}}_{C'} = 0. \tag{71}$$

[8]Out of the infinite possibilities to choose the phases β and γ, it is convenient to pick $\beta = \frac{1}{2}\arg A$ and $\gamma = \frac{1}{2}\arg B$.

subtracting the term which is $\propto C$ from equality a squaring both sides gives

$$\big((A'-B')x+B'\big)^2 = C'^2 x(1-x). \tag{72}$$

This is just a quadratic polynomial in x:

$$\Rightarrow (A'-B')^2x^2 + 2(A'-B')B'x + B^2 = C'^2x - C'^2x^2$$
$$\Rightarrow \underbrace{\big((A'-B')^2 + C'^2\big)}_{M} x^2 + \underbrace{\big(2(A'-B')B' - C'^2\big)}_{N} x + \underbrace{B'^2}_{L} = 0 \tag{73}$$

Let us choose A', B' and C' to be real (we assume that it is always possible). Thus, the phases are

$$\gamma = \frac{1}{2}\arg B, \quad \beta = \arg C - \gamma + \pi m = \arg C - \frac{1}{2}\arg B + \pi m \text{ and also } \beta = \frac{1}{2}\arg A + \frac{\pi}{2}n. \tag{74}$$

Only if the last two expressions for β are consistent, the assumption $\{A', B', C'\} \in \mathbb{R}$ is valid. Thus, one should check that

$$\arg C - \frac{1}{2}\arg A - \frac{1}{2}\arg B = \frac{\pi}{2}l. \tag{75}$$

All is left to determine $|a|^2$ by solving the quadratic polynomial, which now has real coefficients,

$$|a|^2 = \frac{-N + \sqrt{N^2 - 4ML}}{2M}. \tag{76}$$

The properties of the complex-conjugation operator

1. The complex conjugate operator $Kz = \bar{z}$, is an antiunitary operator. This implies that $K(\alpha|a\rangle + \beta|b\rangle) = \bar{\alpha}K|a\rangle + \bar{\beta}K|b\rangle$ and $\langle Ka|Kb\rangle = \langle a|b\rangle^*$.
2. This adjoint of K is defined by $\langle Ka|b\rangle = \langle a|K^\dagger|b\rangle^*$.
3. By definition $K^2 = 1$, thus K^2 is unitary.
4. The adjoint of K is also antiunitary and $KK^\dagger = K^\dagger K = 1$.
   This property should not be confused with the definition of unitary operators, as K is not complex linear and the adjoint of K is defined differently.

   Proof:

$$\langle a|a\rangle^* = \langle Ka|Ka\rangle = \langle a|K^\dagger K|a\rangle \Rightarrow K^\dagger K = 1$$
$$(KK^\dagger)K^2 = K(K^\dagger K)K = K^2 \Rightarrow KK^\dagger = 1.$$

5. The adjoint of K fulfills $K^\dagger = K$.

Proof:

$$K^\dagger = K^\dagger K^2 = (K^\dagger K)K = K \Rightarrow K^\dagger = K.$$

8.2 Thouless representation when zero modes are present

In order to construct the BCS many-body groundstate, we start with the bare vacuum $|0\rangle$ and multiply it by a product of quasi-particle annihilation operators with positive energy. In the end multiply it with half of the zero-mode operators,

$$|\Phi_0\rangle = \prod_i c_{0_i} \prod_j c_{\epsilon_i} |0\rangle. \tag{77}$$

In most cases, there is only one set of zero-modes for which the construed state won't vanishes identically. This point is better understood by considering the Thouless representation of the groundstate,

$$|\Omega\rangle = \sqrt{|\det U|} \exp\left(\sum_{i<j} Z_{ij} \psi_i^\dagger \psi_j^\dagger\right) |0\rangle, \quad Z = (VU^{-1})^*. \tag{78}$$

When the groundstate vanishes identically, $\det U = 0$ and since $\langle 0|\Omega\rangle = \sqrt{|\det U|}$ it means that groundstate is orthogonal to the bare vacuum. In addition, $\det U = 0$ means that U is singular and Z is undefined. Practically, we identify the zero-modes for which $\det U \neq 0$ as the annihilation operators and use them to contract the groundstate.

Constructing the Kitaev chain groundstates (even and odd parity)

The Kitaev chain is a lattice model of a p-wave superconductor in 1D

$$H = -\mu \sum_x c_x^\dagger c_x - \sum_x (t c_x^\dagger c_{x+1} + \Delta c_x c_{x+1} + \text{h.c}), \tag{79}$$

where μ is the chemical potential, t the nearest-neighbor hopping, and Δ the coupling constant.

Let us consider the case $\mu = 0$, $t = \Delta = -1$ for a lattice of 4 sites,

$$H=\sum_{x=1}^{3}(c_x^\dagger c_{x+1}+c_xc_{x+1}+\text{h.c})=\frac{1}{2}\begin{pmatrix}c_1^\dagger\\c_2^\dagger\\c_3^\dagger\\c_4^\dagger\\c_1\\c_2\\c_3\\c_4\end{pmatrix}^T\left(\begin{array}{cccc|cccc}0&1&0&0&0&-1&0&0\\1&0&1&0&1&0&-1&0\\0&1&0&1&0&1&0&-1\\0&0&1&0&0&0&1&0\\\hline 0&1&0&0&0&-1&0&0\\-1&0&1&0&-1&0&-1&0\\0&-1&0&1&0&-1&0&-1\\0&0&-1&0&0&0&-1&0\end{array}\right)\begin{pmatrix}c_1\\c_2\\c_3\\c_4\\c_1^\dagger\\c_2^\dagger\\c_3^\dagger\\c_4^\dagger\end{pmatrix}. \tag{80}$$

Next, we rewrite the fermion operators in terms of Majorana fermion operators,

$$\gamma_x=i\frac{c_x-c_x^\dagger}{\sqrt{2}},\quad \eta_x=\frac{c_x+c_x^\dagger}{\sqrt{2}} \tag{81}$$

This yields

$$H=i\sum_{x=1}^{3}\gamma_x\eta_{x+1}=i\begin{pmatrix}\gamma_1&\gamma_2&\gamma_3&\gamma_4\end{pmatrix}\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\begin{pmatrix}\eta_1\\\eta_2\\\eta_3\\\eta_4\end{pmatrix}. \tag{82}$$

The Hamiltonian eigenstates in the fermion basis are:

$$\begin{array}{cccccccc}\epsilon=0&\epsilon=+1&\epsilon=+1&\epsilon=+1&\epsilon=0&\epsilon=-1&\epsilon=-1&\epsilon=-1\end{array}$$

$$U=\frac{1}{2}\left(\begin{array}{cccc|cccc}1&1&0&0&-1&1&0&0\\0&1&1&0&0&-1&1&0\\0&0&1&1&0&0&-1&1\\1&0&0&1&1&0&0&-1\\\hline -1&1&0&0&1&1&0&0\\0&-1&1&0&0&1&1&0\\0&0&-1&1&0&0&1&1\\1&0&0&-1&1&0&0&1\end{array}\right) \tag{83}$$

$$
\begin{aligned}
\Gamma_{0+} &= c_1 + c_4 - c_1^\dagger + c_4^\dagger \\
\Gamma_{0-} &= -c_1 + c_4 + c_1^\dagger + c_4^\dagger \\
\Gamma_1 &= c_1 + c_2 + c_1^\dagger - c_2^\dagger \\
\Gamma_2 &= c_2 + c_3 + c_2^\dagger - c_3^\dagger \\
\Gamma_3 &= c_3 + c_4 + c_3^\dagger - c_4^\dagger
\end{aligned} \tag{84}
$$

$$
\begin{aligned}
|\Omega\rangle = \Gamma_3\Gamma_2\Gamma_1|0\rangle = \Gamma_3\Gamma_2(|1000\rangle - |0100\rangle) = \Gamma_3(-|1100\rangle + |1010\rangle - |0000\rangle - |0110\rangle) = \\
-|1110\rangle + |1101\rangle - |1000\rangle - |1011\rangle - |0010\rangle + |0001\rangle + |0100\rangle + |0111\rangle
\end{aligned} \tag{85}
$$

$$
\begin{aligned}
\Gamma_{0+}|\Omega\rangle = -|0110\rangle + |1111\rangle + |0101\rangle + |1100\rangle - |0000\rangle + |1001\rangle - |0011\rangle - |1010\rangle + |1010\rangle \\
+ |0011\rangle + |0000\rangle - |1001\rangle - |1100\rangle - |0101\rangle + |0110\rangle - |1111\rangle = 0
\end{aligned} \tag{86}
$$

$$
\begin{aligned}
\Gamma_{0-}|\Omega\rangle = +|0110\rangle + |1111\rangle - |0101\rangle + |1100\rangle + |0000\rangle + |1001\rangle + |0011\rangle - |1010\rangle \\
- |1010\rangle + |0011\rangle + |0000\rangle + |1001\rangle + |1100\rangle - |0101\rangle + |0110\rangle + |1111\rangle = \\
2(|0110\rangle + |1111\rangle - |0101\rangle + |1100\rangle + |0000\rangle + |1001\rangle + |0011\rangle - |1010\rangle)
\end{aligned} \tag{87}
$$

$$
\Rightarrow \Gamma_{0-}|\Omega\rangle = \exp(g_{ij} c_i^\dagger c_j^\dagger), \quad g = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix} \tag{88}
$$

9 The Berry connection

The closed-path Berry phase defined above can be expressed as

$$
\gamma_n = \int_C d\mathbf{R} \cdot \mathcal{A}_n(\mathbf{R}) \tag{89}
$$

where

$$
\mathcal{A}_n(\mathbf{R}) = i\langle n(\mathbf{R})|\nabla_{\mathbf{R}}|n(\mathbf{R})\rangle \tag{90}
$$

is a vector-valued function known as the Berry connection (or Berry potential).

In what follows, we use the expression of the overlap between two different states to derive an explicit expression of the Berry connection.

$$-i\mathcal{A}(\mathbf{R}) \equiv \langle\Phi(\mathbf{R})|\nabla_{\mathbf{R}}|\Phi(\mathbf{R})\rangle = \lim_{\mathbf{R}'\to\mathbf{R}} \nabla_{\mathbf{R}}\langle\Phi(\mathbf{R}')|\Phi(\mathbf{R})\rangle = \lim\left[A(\nabla_{\mathbf{R}}B)S_N \mathrm{pf}\mathcal{Z} + ABS_N\nabla_{\mathbf{R}}\mathrm{pf}\mathcal{Z}\right] \tag{91}$$

where in the last step we used Robledo's formula for the overlap between to states,

$$\langle\Phi(\mathbf{R}')|\Phi(\mathbf{R})\rangle = \underbrace{\sqrt{|\det U(\mathbf{R})|}}_{A}\underbrace{\sqrt{|\det U(\mathbf{R}')|}}_{B}\, S_N \mathrm{pf}\underbrace{\begin{pmatrix} Z(\mathbf{R}) & -1 \\ 1 & -Z^*(\mathbf{R}') \end{pmatrix}}_{\mathcal{Z}}, \quad Z = (VU^{-1})^*. \tag{92}$$

Using the two relations,

$$\partial_x \mathrm{pf}M = \frac{1}{2}\mathrm{pf}M\,\mathrm{tr}(M^{-1}\partial_x M), \quad \partial_x \det M = \det M\,\mathrm{tr}(M^{-1}\partial_x M) \tag{93}$$

we find that

$$\begin{aligned}
-i\mathcal{A}(\mathbf{R}) &= \lim_{\mathbf{R}'\to\mathbf{R}}\left[\frac{B}{4A^3}(\det U^*\ \nabla_{\mathbf{R}}\det U + \det U\ \nabla_{\mathbf{R}}\det U^*)S_N\mathrm{pf}\mathcal{Z} + \frac{1}{2}ABS_N\mathrm{pf}\mathcal{Z}\mathrm{tr}\left(\mathcal{Z}^{-1}\nabla_{\mathbf{R}}\mathcal{Z}\right)\right] \\
&= \frac{1}{2}\left[\mathrm{tr}(U^{-1}\nabla_{\mathbf{R}}U) + \mathrm{tr}(U^{-1}\nabla_{\mathbf{R}}U)^* + \lim_{\mathbf{R}'\to\mathbf{R}}\mathrm{tr}(\mathcal{Z}^{-1}\nabla_{\mathbf{R}}\mathcal{Z})\right]\langle\Phi(\mathbf{R})|\Phi(\mathbf{R})\rangle \\
&= \mathrm{Re}\,\mathrm{tr}(U^{-1}\nabla_{\mathbf{R}}U) + \frac{1}{2}\lim_{\mathbf{R}'\to\mathbf{R}}\mathrm{tr}(\mathcal{Z}^{-1}\nabla_{\mathbf{R}}\mathcal{Z}).
\end{aligned}$$

In order to simplify the expression further, we recall an identity of inverse 2x2 block matrix,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{pmatrix} \tag{94}$$

and substitute $A = Z$,$B = -1$ and $C = 1$,

$$\mathcal{Z}^{-1} = \begin{pmatrix} Z & -1 \\ 1 & D \end{pmatrix}^{-1} = \begin{pmatrix} (Z + D^{-1})^{-1} & (Z + D^{-1})D^{-1} \\ -D^{-1}(Z + D^{-1})^{-1} & D^{-1} - D^{-1}(Z + D^{-1})^{-1}D^{-1} \end{pmatrix}. \tag{95}$$

We continue as follows, multiply the expression for $\mathcal{Z}^{-1}$ by $\nabla_{\mathbf{R}}\mathcal{Z}$, take the trace over the whole expression followed by the limit $D \to Z^\dagger$,

$$\begin{aligned}
\lim_{D\to Z^\dagger}\mathrm{tr}\left(\mathcal{Z}^{-1}_{(i)}\nabla_{\mathbf{R}}\mathcal{Z}\right) &= \lim_{D\to Z^\dagger}\mathrm{tr}\left(\begin{pmatrix} Z & -1 \\ 1 & D \end{pmatrix}^{-1}\begin{pmatrix} Z' & 0 \\ 0 & 0 \end{pmatrix}\right) = \lim_{D\to Z^\dagger}\mathrm{tr}\left((Z + D^{-1})^{-1}Z'\right) \\
&= \mathrm{tr}\left((Z + (Z^\dagger)^{-1})^{-1}Z'\right) = \mathrm{tr}\left((Z + (Z^\dagger)^{-1})^{-1}(Z^\dagger)^{-1}Z^\dagger Z'\right) = \mathrm{tr}\left((1 + Z^\dagger Z)^{-1}Z^\dagger Z'\right).
\end{aligned} \tag{96}$$

Substituting the result in Eq. (94) yields an explicit expression for the Berry connection,

$$-i\mathcal{A}_n(\mathbf{R}) = \langle\Phi(\mathbf{R})|\nabla_{\mathbf{R}}|\Phi(\mathbf{R})\rangle = \mathrm{Re}\,\mathrm{tr}(U^{-1}\nabla_{\mathbf{R}}U) + \frac{1}{2}\mathrm{tr}\left((1+Z^{\dagger}Z)^{-1}Z^{\dagger}Z'\right). \tag{97}$$

The Berry connection in terms of the matrix Z

In order get an expression written solely in terms of Z, we notice that

$$\langle\Phi|\Phi\rangle = 1 \Rightarrow \nabla_{\mathbf{R}}\langle\Phi|\Phi\rangle = 0 \Rightarrow \langle\nabla_{\mathbf{R}}\Phi|\Phi\rangle + \langle\Phi|\nabla_{\mathbf{R}}\Phi\rangle = 0$$
$$\Longrightarrow \langle\Phi|\nabla_{\mathbf{R}}\Phi\rangle = -\langle\nabla_{\mathbf{R}}\Phi|\Phi\rangle, \tag{98}$$

and since

$$\langle\nabla_{\mathbf{R}}\Phi(\mathbf{R})|\Phi(\mathbf{R})\rangle = \lim_{\mathbf{R}'\to\mathbf{R}}\nabla_{\mathbf{R}}\langle\Phi(\mathbf{R})|\Phi(\mathbf{R}')\rangle = \lim\,[(\nabla_{\mathbf{R}}A)BS_N\mathrm{pf}\mathcal{Z} + ABS_N\nabla_{\mathbf{R}}\mathrm{pf}\mathcal{Z}]$$
$$= \mathrm{Re}\,\mathrm{tr}(U^{-1}\nabla_{\mathbf{R}}U) + \frac{1}{2}\lim\mathrm{tr}(\mathcal{Z}^{-1}\nabla_{\mathbf{R}}\mathcal{Z}), \tag{99}$$

the expression $\langle\Phi|\nabla_{\mathbf{R}}\Phi\rangle = \frac{1}{2}(\langle\Phi|\nabla_{\mathbf{R}}\Phi\rangle - \langle\nabla_{\mathbf{R}}\Phi|\Phi\rangle)$ would depend solely on Z. In order to simplify further the expression in Eq. (99), we recall an identity of inverse 2x2 block matrix[11],

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix} \tag{100}$$

and substitute $D = Z^{\dagger}$,$B = -1$ and $C = 1$,

$$\begin{pmatrix} A & -1 \\ 1 & Z^{\dagger} \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} - A^{-1}(Z^{\dagger}+A^{-1})^{-1}A^{-1} & A^{-1}(Z^{\dagger}+A^{-1})^{-1} \\ -(Z^{\dagger}+A^{-1})^{-1}A^{-1} & (Z^{\dagger}+A^{-1})^{-1} \end{pmatrix}. \tag{101}$$

We continue as follows, multiply the expression for $\mathcal{Z}^{-1}$ by $\nabla_{\mathbf{R}}\mathcal{Z}$, take the trace over the whole expression followed by the limit $A \to Z$,

$$\lim_{A\to Z}\mathrm{tr}\left(\mathcal{Z}_{(i)}^{-1}\nabla_{\mathbf{R}}\mathcal{Z}\right) = \lim_{A\to Z}\mathrm{tr}\left(\begin{pmatrix} A & -1 \\ 1 & Z^{\dagger} \end{pmatrix}^{-1}\begin{pmatrix} 0 & 0 \\ 0 & Z'^{\dagger} \end{pmatrix}\right) = \lim_{A\to Z}\mathrm{tr}\left((Z^{\dagger}+A^{-1})^{-1}Z'^{\dagger}\right) =$$
$$\mathrm{tr}\left((Z^{\dagger}+Z^{-1})^{-1}Z'^{\dagger}\right) = \mathrm{tr}\left((ZZ^{-1})(Z^{\dagger}+Z^{-1})^{-1}Z'^{\dagger}\right) = \mathrm{tr}\left((1+Z^{\dagger}Z)^{-1}Z'^{\dagger}Z\right), \tag{102}$$

where in last step we exploited the fact that the trace is invariant under cyclic permutations, $\mathrm{tr}(A_1, A_2, ..., A_N) = \mathrm{tr}(A_N, A_1, ..., A_{N-1})$ for any $N \in \mathbb{N}$ and the property of invertible matrices, $(AB)^{-1} = B^{-1}A^{-1}$. Substituting the result in Eq. (99) yields

$$\langle\nabla_{\mathbf{R}}\Phi(\mathbf{R})|\Phi(\mathbf{R})\rangle = \mathrm{Re}\,\mathrm{tr}(U^{-1}\nabla_{\mathbf{R}}U) + \frac{1}{2}\mathrm{tr}\left((1+Z^{\dagger}Z)^{-1}Z'^{\dagger}Z\right). \tag{103}$$

Combining Eq.(97) and Eq.(103) yields

$$\langle\Phi|\nabla_{\mathbf{R}}\Phi\rangle = \frac{1}{2}(\langle\Phi|\nabla_{\mathbf{R}}\Phi\rangle - \langle\nabla_{\mathbf{R}}\Phi|\Phi\rangle) = \frac{1}{4}\mathrm{tr}\left((1+Z^\dagger Z)^{-1}(Z^\dagger Z' - Z'^\dagger Z)\right) \tag{104}$$

which is the celebrated expression for the Berry connection appearing in Read's article[12].

Numerical calculations of the Berry connection

Here we write the berry connection in terms of the matrices U, V and U^{-1}. In addition, we use a few algebraic manipulations so the expression includes only derivatives of U and V.

We first recall that the unitary of the eigenstates matrix gives the following identities:

$$\begin{pmatrix} U^\dagger & V^\dagger \\ V^T & U^T \end{pmatrix}\begin{pmatrix} U & \bar{V} \\ V & \bar{U} \end{pmatrix} = 1 \quad \Rightarrow \quad \begin{matrix} U^\dagger U + V^\dagger V = 1 & V^T\bar{V} + U^T\bar{U} = 1 \\ U^\dagger\bar{V} + V^\dagger\bar{U} = 0 & V^T U + U^T V = 0 \end{matrix}$$

$$\begin{pmatrix} U & \bar{V} \\ V & \bar{U} \end{pmatrix}\begin{pmatrix} U^\dagger & V^\dagger \\ V^T & U^T \end{pmatrix} = 1 \quad \Rightarrow \quad \begin{matrix} UU^\dagger + \bar{V}V^T = 1 & VV^\dagger + \bar{U}U^T = 1 \\ UV^\dagger + \bar{V}U^T = 0 & VU^\dagger + \bar{U}V^T = 0 \end{matrix}$$

Our starting point is an expression of the berry connection [12] in terms of matrix Z,

$$i\langle\Phi|\nabla_{\mathbf{R}}\Phi\rangle = \frac{i}{4}\mathrm{tr}\left[\left(1+Z^\dagger Z\right)^{-1}\left(Z^\dagger Z' - \left(Z^\dagger\right)' Z\right)\right], \quad Z = (VU^{-1})^* \tag{105}$$

Next, we rewrite $\left(1+Z^\dagger Z\right)^{-1}$ in term of U:

$$\begin{aligned} \left[\left(1+Z^\dagger Z\right)^{-1}\right]^* &= \left(1+(VU^{-1})^\dagger(VU^{-1})\right)^{-1} = \left(1+(U^\dagger)^{-1}V^\dagger VU^{-1}\right)^{-1} \\ &= \left((U^\dagger)^{-1}\underbrace{\left(U^\dagger U + V^\dagger V\right)}_{I}U^{-1}\right)^{-1} = UU^\dagger. \end{aligned} \tag{106}$$

Using this result we calculate the first term in the difference:

$$\begin{aligned} \mathrm{tr}\left[\left(1+Z^\dagger Z\right)^{-1} Z^\dagger Z'\right]^* &= \mathrm{tr}\left[UU^\dagger(U^\dagger)^{-1}V^\dagger\left(V'U^{-1} + V(U^{-1})'\right)\right] \\ &= \mathrm{tr}\left(V^\dagger V'\right) - \mathrm{tr}\left(V^\dagger VU^{-1}U'\right), \end{aligned} \tag{107}$$

where in the last step we "moved" the derivative from U^{-1} to U using the relation

$$I' = (U^{-1}U)' = (U^{-1})'U + U^{-1}U' \Rightarrow (U^{-1})'U = -U^{-1}U' \tag{108}$$

Using the result in Eq.(106) we calculate the first term in the difference:

$$\begin{aligned}
\mathrm{tr}\left[\left(1+Z^\dagger Z\right)^{-1}\left(Z^\dagger\right)' Z\right]^* &= \mathrm{tr}\left[UU^\dagger\left[\left(\left(U^\dagger\right)^{-1}\right)V^\dagger\right]' VU^{-1}\right] \\
&= \mathrm{tr}\left[VU^\dagger\left[\left(\left(U^\dagger\right)^{-1}\right)' V^\dagger + \left(U^\dagger\right)^{-1}\left(V^\dagger\right)'\right]\right] = \\
&= \mathrm{tr}\left(V\left(V^\dagger\right)'\right) - \mathrm{tr}\left[\left(U^\dagger\right)^{-1} V^\dagger V\left(U^\dagger\right)'\right],
\end{aligned} \tag{109}$$

where in the last step we again "moved" the derivative from U^{-1} to U using the relation

$$(U^{-1})'U = -U^{-1}U' \Rightarrow U^\dagger\left[\left(U^\dagger\right)^{-1}\right]' = -\left(U^\dagger\right)'\left(U^\dagger\right)^{-1} \tag{110}$$

combing Eq.(107) and Eq.(109) the Berry connection can be written as

$$i\langle\Phi|\nabla_{\mathbf{R}}\Phi\rangle = \frac{i}{4}\mathrm{tr}\left[V^\dagger V' - V\left(V^\dagger\right)' + \left(U^\dagger\right)^{-1} V^\dagger V\left(U^\dagger\right)' - V^\dagger V U^{-1} U'\right]^* \tag{111}$$

10 The generalized density matrix

The generalized density matrix is defined as

$$\mathcal{R} = \begin{pmatrix} \langle\Phi|\psi_m^\dagger\psi_n|\Phi\rangle & \langle\Phi|\psi_m\psi_n|\Phi\rangle \\ \langle\Phi|\psi_m^\dagger\psi_n^\dagger|\Phi\rangle & \langle\Phi|\psi_m\psi_n^\dagger|\Phi\rangle \end{pmatrix} = \begin{pmatrix} \rho_{n,m} & \kappa_{n,m} \\ -\bar{\kappa}_{n,m} & 1-\bar{\rho}_{n,m}, \end{pmatrix} \tag{112}$$

where ρ and κ are called the *normal* and *abnormal* density (or density matrix and pairing tensor). Furthermore, the generalized density matrix is Hermitian, idempotent ($\mathcal{R} = \mathcal{R}^2$) and $\mathcal{R}$ admits only two eigenvalues: 0 and 1. Eigenstates that correspond to eigenvalues zero and one are also eigenstates of the single particle Hamiltonian, H with positive and negative energies, respectively.

In this section we derive simple expressions for the correlators k_{ij} and ρ_{ij}, in terms of the skew-symmetric matrix, $Z = \overline{VU^{-1}}$. We begin by pointing out that the derivative of the *unnormalized* many-body states with respect to a matrix element of Z yields:

$$\begin{aligned} \partial_{Z_{m,n}}|\phi\rangle &= \frac{1}{2}\left(\psi_m^\dagger\psi_n^\dagger - \psi_n^\dagger\psi_m^\dagger\right)|\phi\rangle = \psi_m^\dagger\psi_n^\dagger|\phi\rangle \\ \partial_{\bar{Z}_{m,n}}|\phi\rangle &= \langle\phi|\frac{1}{2}\left(\psi_n\psi_m - \psi_m\psi_n\right) = \langle\phi|\psi_n\psi_m. \end{aligned} \tag{113}$$

Thus, the correlators can be expressed as

$$\begin{aligned} \kappa_{m,n} =& \langle\Phi(Z)|\psi_n\psi_m|\Phi(Z)\rangle = \lim_{\tilde{Z}\to Z}\partial_{\bar{Z}_{m,n}}\log\langle\Phi(Z)|\Phi(\tilde{Z})\rangle \\ \bar{\kappa}_{m,n} =& \langle\Phi(Z)|\psi_m^\dagger\psi_n^\dagger|\Phi(Z)\rangle = \lim_{\tilde{Z}\to Z}\partial_{Z_{m,n}}\log\langle\Phi(\tilde{Z})|\Phi(Z)\rangle. \end{aligned} \tag{114}$$

A many-body state, $\Phi(Z)\rangle$ is required to vanish when a quasiparticle annihilation operator, c_ϵ acts on it,

$$c_\epsilon|\Phi(Z)\rangle = 0, \tag{115}$$

for any ϵ belonging to a certain set of single-particle eigenenergies, which characterize the $|\Phi(Z)\rangle$. The annihilation operators take the form

$$c_\epsilon = U_{\epsilon i}^\dagger\psi_i + V_{\epsilon i}^\dagger\psi_i^\dagger, \tag{116}$$

where U_ϵ and V_ϵ are the particle and hole parts of a single-particle eigenstate with eigenenergy ϵ, respectively. However, we can use a canonical transformation to get a new set of quasiparticle annihilation operators, $\tilde{c}_i$ that depend solely on Z,

$$(U^{-1})^\dagger c = (U^{-1})^\dagger U^\dagger\psi + \left(VU^{-1}\right)^\dagger\psi^\dagger \Rightarrow \tilde{c} = \psi - Z\psi^\dagger, \tag{117}$$

where we used the Nambu spinors $c^T = (c_{\epsilon_1}, c_{\epsilon_2}, \ldots, c_{\epsilon_N})$, $\psi^T = (\psi_1, \psi_2, \ldots, \psi_N)$ and $\tilde{c}^T = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_N)$. Since the transformation is only among the quasiparticle annihilation operators, the operators c and $\tilde{c}$ share the same many-body state and thus,

$$\left(\psi - Z\psi^\dagger\right)|\Phi(Z)\rangle = 0. \tag{118}$$

Using our new set of quasiparticle annihilation operators, $\tilde{c}$, we manipulate the expression $\Phi|\tilde{c}_i^\dagger \tilde{c}_j|\Phi\rangle = 0$ to get a relation between κ and ρ in terms of Z:

$$\langle\Phi|\left(\psi_j^\dagger - \bar{Z}_{ji}\psi_i\right)\left(\psi_k - Z_{kl}\psi_l^\dagger\right)|\Phi\rangle = \\ \langle\Phi|\left(\psi_j^\dagger\psi_k - Z_{kl}\psi_j^\dagger\psi_l^\dagger - \bar{Z}_{ji}\psi_i\psi_k + \bar{Z}_{ji}Z_{kl}\psi_i\psi_l^\dagger\right)|\Phi\rangle = \langle\Phi|\psi_j^\dagger\psi_k^\dagger - Z_{k,l}\psi_j^\dagger\psi_l^\dagger|\Phi\rangle = 0 \quad (119)$$

where we used the commutation relation in Eq.(13),

$$\left[\psi_k, e^{\frac{1}{2}Z_{m,n}\psi_m^\dagger\psi_n^\dagger}\right] = e^{\frac{1}{2}Z_{m,n}\psi_m^\dagger\psi_n^\dagger}Z_{k,l}\psi_l^\dagger \Rightarrow -\bar{Z}_{ij}\psi_i\psi_k|\Phi\rangle = -\bar{Z}_{i,j}\psi_i e^{\frac{1}{2}Z_{mn,}\psi_m^\dagger\psi_n^\dagger}\left(\psi_k + Z_{k,l}\psi_l^\dagger\right)|0\rangle \\ = -\bar{Z}_{i,j}Z_{k,l}\psi_i\psi_l^\dagger|\Phi\rangle. \quad (120)$$

Thus, the relation between the normal density ρ and the anomalous density κ is

$$\rho_{k,j} = \langle\Phi|\psi_j^\dagger\psi_k|\Phi\rangle = Z_{k,l}\langle\Phi|\psi_j^\dagger\psi_l^\dagger|\Phi\rangle = Z_{k,l}\kappa_{l,j}^\dagger \Rightarrow \rho = Z\kappa^\dagger = -Z\bar{\kappa} = \kappa Z^\dagger = -\kappa\bar{Z}, \quad (121)$$

where we used the properties $\rho^\dagger = \rho$, $\kappa = -\kappa^T$ and $Z = -Z^T$. Based on the identity in Eq.(133), we derive a simple expression for κ in terms of Z:

$$\bar{k}_{m,n} = \frac{\langle\phi|\partial_{Z_{m,n}}\phi\rangle}{\langle\phi|\phi\rangle} = \frac{1}{2}\mathrm{tr}\left[\left(1 + Z^\dagger Z\right)^{-1} Z^\dagger \partial_{z_{m,n}} Z\right] \\ = \frac{1}{2}\left(\left[\left(1 + Z^\dagger Z\right)^{-1} Z^\dagger\right]_{n,m} - \left[\left(1 + Z^\dagger Z\right)^{-1} Z^\dagger\right]_{m,n}\right) = -\left[\left(1 + Z^\dagger Z\right)^{-1} Z^\dagger\right]_{m,n}, \quad (122)$$

where we exploited the skew-symmetry of $\left(1 + Z^\dagger Z\right)^{-1} Z^\dagger$. In order to prove that indeed $\left(1 + Z^\dagger Z\right)^{-1} Z^\dagger$ is skew-symmetric we substitute $Z = \overline{VU^{-1}}$,

$$\left(1 + Z^\dagger Z\right)^{-1} Z^\dagger = \left(1 + (VU^{-1})^T \bar{V}\bar{U}^{-1}\right)^{-1}(VU^{-1})^T = \left(1 + (U^{-1})^T V^T \bar{V}\bar{U}^{-1}\right)^{-1}(U^{-1})^T V^T \\ = \left((U^{-1})^T\left(U^T\bar{U} + V^T\bar{V}\right)\bar{U}^{-1}\right)^{-1}(U^{-1})^T V^T = \bar{U}U^T(U^{-1})^T V^T = \bar{U}V^T, \quad (123)$$

and using the identity $P^\dagger P = 1$ we find that

$$(PP^\dagger)_{1,2} = 0 \Rightarrow UV^\dagger + \bar{V}U^T = 0 \Rightarrow \bar{U}V^T = -(\bar{U}V^T)^T. \quad (124)$$

Finally, it follows from the derivation above that

$$\kappa = -UV^\dagger \quad \rho = -Z\bar{\kappa} = \bar{V}V^T. \quad (125)$$

In what follows, we use the expression of the overlap between two different states to derive an expression for the anomalous density matrix in terms of the Z matrix.

$$\langle\phi(Z)|\partial_{Z_{m,n}}\phi(Z)\rangle = \lim_{\tilde{Z}\to Z}\partial_{Z_{m,n}}\langle\phi(\tilde{Z})|\phi(Z)\rangle = \lim_{\tilde{Z}\to Z} S_N \nabla_{\mathbf{R}}\mathrm{pf}\mathcal{Z} \tag{126}$$

where in the last step we used Robledo's formula for the overlap between to states,

$$\langle\phi(\tilde{Z})|\phi(Z)\rangle = S_N \mathrm{pf}\underbrace{\begin{pmatrix} Z & -1 \\ 1 & -\bar{Z}\end{pmatrix}}_{\mathcal{Z}}, \quad Z = \overline{VU^{-1}}. \tag{127}$$

Using the relation,

$$\partial_x \mathrm{pf}M = \frac{1}{2}\mathrm{pf}M\,\mathrm{tr}(M^{-1}\partial_x M), \tag{128}$$

we find that

$$\langle\phi(Z)|\partial_{Z_{m,n}}\phi(Z)\rangle = \lim_{\tilde{Z}\to Z}\frac{1}{2}S_N\mathrm{pf}\mathcal{Z}\,\mathrm{tr}\left(\mathcal{Z}^{-1}\nabla_{\mathbf{R}}\mathcal{Z}\right) = \lim_{\tilde{Z}\to Z}\frac{1}{2}\mathrm{tr}(\mathcal{Z}^{-1}\nabla_{\mathbf{R}}\mathcal{Z})\langle\phi(\tilde{Z})|\phi(Z)\rangle \tag{129}$$

In order to simplify the expression further, we recall an identity of inverse 2x2 block matrix,

$$\begin{pmatrix} A & B \\ C & D\end{pmatrix}^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & -(A-BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & D^{-1}+D^{-1}C(A-BD^{-1}C)^{-1}BD^{-1}\end{pmatrix} \tag{130}$$

and substitute $A = Z$, $B = -1$, $C = 1$ and $D = \tilde{Z}^{\dagger}$

$$\mathcal{Z}^{-1} = \begin{pmatrix} Z & -1 \\ 1 & D\end{pmatrix}^{-1} = \begin{pmatrix} (Z+\tilde{Z}^{\dagger-1})^{-1} & (Z+\tilde{Z}^{\dagger-1})\tilde{Z}^{\dagger-1} \\ -\tilde{Z}^{\dagger-1}(Z+\tilde{Z}^{\dagger-1})^{-1} & \tilde{Z}^{\dagger-1}-\tilde{Z}^{\dagger-1}(Z+\tilde{Z}^{\dagger-1})^{-1}\tilde{Z}^{\dagger-1}\end{pmatrix}. \tag{131}$$

We continue as follows, multiply the expression for $\mathcal{Z}^{-1}$ by $\partial_{Z_{m,n}}\mathcal{Z}$, take the trace over the whole expression followed by the limit $\tilde{Z}\to Z$,

$$\begin{aligned}\lim_{\tilde{Z}\to Z}\mathrm{tr}\left(\mathcal{Z}_{(i)}^{-1}\nabla_{\mathbf{R}}\mathcal{Z}\right) &= \lim_{\tilde{Z}\to Z}\mathrm{tr}\left(\begin{pmatrix} Z & -1 \\ 1 & \tilde{Z}^{\dagger}\end{pmatrix}^{-1}\begin{pmatrix} Z' & 0 \\ 0 & 0\end{pmatrix}\right) = \lim_{\tilde{Z}\to Z}\mathrm{tr}\left((Z+(\tilde{Z}^{\dagger})^{-1})^{-1}Z'\right) \\ &= \mathrm{tr}\left((Z+(Z^{\dagger})^{-1})^{-1}Z'\right) = \mathrm{tr}\left((Z+(Z^{\dagger})^{-1})^{-1}(Z^{\dagger})^{-1}Z^{\dagger}Z'\right) = \mathrm{tr}\left((1+Z^{\dagger}Z)^{-1}Z^{\dagger}Z'\right).\end{aligned} \tag{132}$$

Substituting the result in Eq. (129) yields an explicit expression for the anomalous density matrix,

$$\bar{\kappa}_{m,n} = \langle\Phi(Z)|\psi_m^{\dagger}\psi_n^{\dagger}|\Phi(Z)\rangle = \frac{\langle\phi(Z)|\partial_{Z_{m,n}}\phi(Z)\rangle}{\langle\phi(Z)|\phi(Z)\rangle} = \frac{1}{2}\mathrm{tr}\left((1+Z^{\dagger}Z)^{-1}Z^{\dagger}\partial_{Z_{m,n}}Z\right). \tag{133}$$

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Appendix E

The Bogoliubov-de Gennes models on a lattice

Contents

1 The s-wave superconductor

1.1 s-wave superconductor - continuum model

The s-wave continuum Hamiltonian:

$$\mathcal{H} = \begin{pmatrix} \psi^\dagger_{\mathbf{x},\uparrow} & \psi_{\mathbf{x},\downarrow} \end{pmatrix} \begin{pmatrix} \frac{1}{2m}(-\mathbf{p}+\mathbf{A})^2 - \mu & \Delta \\ \overline{\Delta} & -\frac{1}{2m}(\mathbf{p}+\mathbf{A})^2 + \mu \end{pmatrix} \begin{pmatrix} \psi_{\mathbf{x},\uparrow} \\ \psi^\dagger_{\mathbf{x},\downarrow} \end{pmatrix} \tag{1}$$

We note that, in contrary to p-wave superconductor, the Bogoliubov representation of the s-wave continuum Hamiltonian does not incorporate a factor of $1/2$.

1.2 s-wave superconductor - tight binding model on a square lattice

$$\mathcal{H} = \sum_{\mathbf{k},\sigma} \mathbf{c}^{\dagger}_{\mathbf{k},\sigma} \left(\frac{\mathbf{k}^2}{2m} - \mu \right) \mathbf{c}_{\mathbf{k},\sigma} + \sum_{\mathbf{k}} \left(\Delta \mathbf{c}^{\dagger}_{\mathbf{k}\uparrow} \mathbf{c}^{\dagger}_{-\mathbf{k}\downarrow} + \bar{\Delta} \mathbf{c}_{-\mathbf{k}\downarrow} \mathbf{c}_{\mathbf{k}\uparrow} \right) \tag{2}$$

We transform from the the continuum to a square lattice using the approximation $\frac{k^2a^2}{2} \approx 1 - \cos ka$ and defining $t \equiv \frac{1}{2ma^2}$ (for simplicity we assume $a = 1$):

$$\begin{aligned}
\mathcal{H} &= \sum_{\mathbf{k},\sigma} \mathbf{c}^{\dagger}_{\mathbf{k},\sigma} \left(-\frac{\cos k_x + cosk_y}{m} - \mu + \frac{2}{m} \right) \mathbf{c}_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} \left(\Delta \mathbf{c}^{\dagger}_{\mathbf{k}\uparrow} \mathbf{c}^{\dagger}_{-\mathbf{k}\downarrow} + \bar{\Delta} \mathbf{c}_{-\mathbf{k}\downarrow} \mathbf{c}_{\mathbf{k}\uparrow} \right) \\
&= \sum_{\mathbf{k},\sigma} \mathbf{c}^{\dagger}_{\mathbf{k},\sigma} \left(-2t(\cos k_x + cosk_y) - \mu + 4t \right) \mathbf{c}_{\mathbf{k}\sigma} + \sum_{\mathbf{k}} \left(\Delta \mathbf{c}^{\dagger}_{\mathbf{k}\uparrow} \mathbf{c}^{\dagger}_{-\mathbf{k}\downarrow} + \bar{\Delta} \mathbf{c}_{-\mathbf{k}\downarrow} \mathbf{c}_{\mathbf{k}\uparrow} \right) \\
&= \sum_{\mathbf{k},\sigma} \mathbf{c}^{\dagger}_{\mathbf{k},\sigma} \left(-t(\cos k_x + \cos k_y) - \frac{\mu}{2} + 2t \right) \mathbf{c}_{\mathbf{k}\sigma} - \mathbf{c}_{-\mathbf{k},\sigma} \left(-t(\cos k_x + \cos k_y) - \frac{\mu}{2} + 2t \right) \mathbf{c}^{\dagger}_{-\mathbf{k}\sigma} \\
&\quad + \sum_{\mathbf{k}} \frac{1}{2} \left(\Delta \mathbf{c}^{\dagger}_{\mathbf{k}\uparrow} \mathbf{c}^{\dagger}_{-\mathbf{k}\downarrow} - \Delta \mathbf{c}^{\dagger}_{\mathbf{k}\downarrow} \mathbf{c}^{\dagger}_{-\mathbf{k}\uparrow} + \bar{\Delta} \mathbf{c}_{-\mathbf{k}\downarrow} \mathbf{c}_{\mathbf{k}\uparrow} - \bar{\Delta} \mathbf{c}_{-\mathbf{k}\uparrow} \mathbf{c}_{\mathbf{k}\downarrow} \right) + \underbrace{\sum_{\mathbf{k},\sigma} \left(-t(\cos k_x + \cos k_y) - \frac{\mu}{2} + 2t \right)}_{E_0}
\end{aligned} \tag{3}$$

In term on Nambu spinors the Hamiltonian is

$$\mathcal{H}(\mathbf{k}) = \frac{1}{2} \sum_{\mathbf{k}} \begin{pmatrix} \mathbf{c}^{\dagger}_{\mathbf{k}\uparrow} & \mathbf{c}^{\dagger}_{\mathbf{k}\downarrow} & \mathbf{c}_{-\mathbf{k}\uparrow} & \mathbf{c}_{-\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} \epsilon(\mathbf{k}) & 0 & 0 & \Delta \\ 0 & \epsilon(\mathbf{k}) & -\Delta & 0 \\ 0 & -\bar{\Delta} & -\epsilon(-\mathbf{k}) & 0 \\ \bar{\Delta} & 0 & 0 & -\epsilon(-\mathbf{k}) \end{pmatrix} \begin{pmatrix} \mathbf{c}_{\mathbf{k}\uparrow} \\ \mathbf{c}_{\mathbf{k}\downarrow} \\ \mathbf{c}^{\dagger}_{-\mathbf{k}\uparrow} \\ \mathbf{c}^{\dagger}_{-\mathbf{k}\downarrow} \end{pmatrix} \tag{4}$$

where $\epsilon(\mathbf{k}) = -2t(\cos k_x + \cos k_y) - \mu + 4t$. The transformation of the fermion operators form the momentum basis to the spatial basis is

$$\mathbf{c}_k = \sum_x \langle k|x \rangle \psi_x = \frac{1}{\sqrt{N}} \sum_x e^{-ikx} \psi_x \tag{5}$$

where N is the number of lattice sites, $k_n = \frac{2\pi n}{Na}$, $x_m = ma$ and $0 < m \leq N$. The momentum states form an orthogonal set:

$$\sum_k \exp\left(ik(x_i - x_j)\right) = \sum_{n=1}^{N} \left[\exp\left(\frac{i2\pi a(i-j)}{Na} \right) \right]^n = \begin{cases} N, & i = j \\ \frac{1-e^{2\pi(i-j)}}{e^{-\frac{i2\pi(i-j)}{N}}-1} & i \neq j \end{cases} = N\delta_{i,j}$$

Using the last two properties we find

$$\sum_k e^{ika} \mathbf{c}^{\dagger}_k \mathbf{c}_k = \frac{1}{N} \sum_{k,m,n} e^{ik(m+1-m')a} \mathbf{c}^{\dagger}_m \mathbf{c}_{m'} = \sum_{m,m'} \delta_{m+1,m'} \mathbf{c}^{\dagger}_m \mathbf{c}_{m'} = \sum_m \mathbf{c}^{\dagger}_m \mathbf{c}_{m+1} \tag{6}$$

and

$$\sum_k \cos ka \mathbf{c}^{\dagger}_k \mathbf{c}_k = \frac{1}{2} \sum_k \left(e^{ika} + e^{-ika} \right) \mathbf{c}^{\dagger}_k \mathbf{c}_k = \frac{1}{2} \sum_m \left(\psi^{\dagger}_{m+1} \psi_m + \psi^{\dagger}_m \psi_{m+1} \right). \tag{7}$$

From here it is straight forward to get the s-wave Hamiltonian in the tight-binding model,

$$\mathcal{H} = \sum_{m,n,\sigma} \Bigg[-t \left(\psi^{\dagger}_{m+1,n,\sigma} \psi_{m,n,\sigma} + \psi^{\dagger}_{m,n,\sigma} \psi_{m+1,n,\sigma} + \psi^{\dagger}_{m,n+1,\sigma} \psi_{m,n,\sigma} + \psi^{\dagger}_{m,n,\sigma} \psi_{m,n+1,\sigma} \right) = \\ -(\mu - 4t) \psi^{\dagger}_{m,n,\sigma} \psi_{m,n,\sigma} + \Delta \psi^{\dagger}_{m,n,\uparrow} \psi^{\dagger}_{m,n,\downarrow} + \bar{\Delta} \psi_{m,n,\downarrow} \psi_{m,n,\uparrow} \Bigg] \tag{8}$$

2 The p-wave superconductor tight-binding model

2.1 Transforming from a p-wave superconductor tight-binding model of a square lattice to the continuum limit

We begin with the p-wave lattice Hamiltonian,

$$\mathcal{H} = \sum_{m,n}\left[-t\left(\psi^\dagger_{m+1,n}\psi_{m,n} + \psi^\dagger_{m,n+1}\psi_{m,n}\right) - \frac{1}{2}(\mu - 4t)\psi^\dagger_{m,n}\psi_{m,n} + \Delta\left(\psi^\dagger_{m+1,n}\psi^\dagger_{m,n} + i\psi^\dagger_{m,n+1}\psi^\dagger_{m,n}\right)\right] + \text{h.c.} \quad (9)$$

Our next step is to transform the Hamiltonian into momentum space. The Fourier transform of the creation and annihilation operators combinations that appear in the Hamiltonian can be written as

$$\psi_{m,n} = \frac{1}{\sqrt{N}}\sum_{k_x,k_y} e^{ik_x ma + ik_y na}\mathbf{c}_{k_x,k_y} \quad (10)$$

where for a lattice size (q_x, q_y) we have $N = q_x \times q_y$.
A translation operator transforms as

$$\sum_{m,n}\psi^\dagger_{m+1,n}\psi_{m,n} = \frac{1}{N}\sum_{m,n}\sum_{k_x,k_y}\sum_{\tilde{k_x},\tilde{k_y}} e^{ik_x(m+1)a+ik_y na}e^{-i\tilde{k}_x ma - i\tilde{k}_y na}\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{\tilde{k}_x,\tilde{k}_y} =$$
$$= \frac{1}{N}\sum_{m,n}\sum_{k_x,k_y}\sum_{\tilde{k_x},\tilde{k_y}} e^{ima(k_x-\tilde{k}_x)}e^{ina(k_y-\tilde{k}_y)}e^{iak_x}\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{\tilde{k}_x,\tilde{k}_y} =$$
$$= \sum_{k_x,k_y}\sum_{\tilde{k_x},\tilde{k_y}}\delta_{\tilde{k}_x,\tilde{k}_x}\delta_{\tilde{k}_y,\tilde{k}_y}e^{iak_x}\mathbf{c}^\dagger_{k_x,k_y}\psi_{\tilde{k}_x,\tilde{k}_y} = \sum_{k_x,k_y}e^{iak_x}\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{k_x,k_y} \quad (11)$$

where assumed periodic boundary conditions, $qa(\tilde{k} - k) = 2\pi p$ with $p \in \mathbb{N}$ in order to get an expression for the Kronecker delta -

$$\sum_{m=1}^{q} e^{ima(\tilde{k}-k)} = \sum_{m=1}^{q}\left(e^{ia(\tilde{k}-k)}\right)^m = \begin{cases} q & ,\tilde{k} = k \\ \frac{e^{iqa(\tilde{k}-k)}-1}{1-e^{-ia(\tilde{k}-k)}} = 0 & ,\tilde{k} \neq k \end{cases} = q\delta_{\tilde{k},k} \quad (12)$$

and the conjugate transpose of the last combination yields -

$$\sum_{m,n}\psi^\dagger_{m,n}\psi_{m+1,n} = \sum_{k_x,k_y} e^{-iak_x}\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{k_x,k_y} \quad (13)$$

The sum of the translation operators in the x direction gives

$$\sum_{m,n}\left(\psi^\dagger_{m+1,n}\psi_{m,n} + \psi^\dagger_{m,n}\psi_{m+1,n}\right) = \sum_{k_x,k_y}(e^{iak_x} + e^{-iak_x})\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{k_x,k_y} = 2\sum_{k_x,k_y}\cos(ak_x)\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{k_x,k_y} \quad (14)$$

Similarly, for the y direction we have

$$\sum_{m,n} \psi^\dagger_{m,n+1}\psi_{m,n} = \sum_{k_x,k_y} e^{iak_y}\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{k_x,k_y} \tag{15}$$

and its conjugate transpose

$$\sum_{m,n} \psi^\dagger_{m,n}\psi_{m,n+1} = \sum_{k_x,k_y} e^{-iak_y}\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{k_x,k_y} \tag{16}$$

The sum of the translation operators in the y direction gives

$$\sum_{m,n} \left(\psi^\dagger_{m,n+1}\psi_{m,n} + \psi^\dagger_{m,n}\psi_{m,n+1}\right) = \sum_{k_x,k_y} (e^{iak_y} + e^{-iak_y})\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{k_x,k_y} = 2\sum_{k_x,k_y} \cos(ak_y)\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{k_x,k_y} \tag{17}$$

The occupation number operator is

$$\sum_{m,n} \psi^\dagger_{m,n}\psi_{m,n} = \sum_{k_x,k_y} \mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{k_x,k_y} \tag{18}$$

The interaction terms are

$$\begin{aligned}\sum_{m,n} \psi^\dagger_{m+1,n}\psi^\dagger_{m,n} &= \sum_{k_x,k_y} e^{iak_x}\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}^\dagger_{-k_x,-k_y} = \frac{1}{2}\sum_{k_x,k_y} e^{iak_x}\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}^\dagger_{-k_x,-k_y} + \frac{1}{2}\sum_{-k_x,-k_y} e^{-iak_x}\mathbf{c}^\dagger_{-k_x,-k_y}\mathbf{c}^\dagger_{k_x,k_y} \\ &= \frac{1}{2}\sum_{k_x,k_y} e^{iak_x}\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}^\dagger_{-k_x,-k_y} - \frac{1}{2}\sum_{k_x,k_y} e^{-iak_x}\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}^\dagger_{-k_x,-k_y} = i\sum_{k_x,k_y} \sin(ak_x)\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}^\dagger_{-k_x,-k_y}\end{aligned} \tag{19}$$

In the last derivation we used the fact that the sum is over a symmetric range and the fermionic commutation relations of creation and annihilation operators. Similarly, we find that

$$\sum_{m,n} \psi_{m,n}\psi_{m+1,n} = \sum_{k_x,k_y} e^{-iak_x}\mathbf{c}_{-k_x,-k_y}\mathbf{c}_{k_x,k_y} = -i\sum_{k_x,k_y} \sin(ak_x)\mathbf{c}_{-k_x,-k_y}\mathbf{c}_{k_x,k_y} \tag{20}$$

$$\sum_{m,n} \psi^\dagger_{m,n+1}\psi^\dagger_{m,n} = \sum_{k_x,k_y} e^{iak_y}\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}^\dagger_{-k_x,-k_y} = i\sum_{k_x,k_y} \sin(ak_y)\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}^\dagger_{-k_x,-k_y} \tag{21}$$

$$\sum_{m,n} \psi_{m,n}\psi_{m,n+1} = \sum_{k_x,k_y} e^{-iak_y}\mathbf{c}_{-k_x,-k_y}\mathbf{c}_{k_x,k_y} = -i\sum_{k_x,k_y} \sin(ak_y)\mathbf{c}_{-k_x,-k_y}\mathbf{c}_{k_x,k_y} \tag{22}$$

Finally, we can write the transformed p-wave Hamiltonian as

$$\begin{aligned}\mathcal{H} = \sum_{k_x,k_y} \Big\{&-2t\cos(ak_x)\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{k_x,k_y} - 2t\cos(ak_y)\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{k_x,k_y} - (\mu - 4t)\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{k_x,k_y} \\ &+ \left(i\Delta\sin(ak_x)\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}^\dagger_{-k_x,-k_y} - i\Delta^*\sin(ak_x)\mathbf{c}_{-k_x,-k_y}\mathbf{c}_{k_x,k_y}\right) + \\ &\left(i\Delta[i\sin(ak_y)]\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}^\dagger_{-k_x,-k_y} + i\Delta^*[i\sin(ak_y)]\mathbf{c}_{-k_x,-k_y}\mathbf{c}_{k_x,k_y}\right)\Big\}\end{aligned} \tag{23}$$

In terms of Nambu spinors it can be written (up to an additive constant) as

$$\mathcal{H} = \sum_{k_x,k_y} \Psi^\dagger_{\mathbf{k}} \begin{pmatrix} -t\left(\cos(ak_x)+\cos(ak_y)\right) - (\frac{\mu}{2} - 2t) & i\Delta\left(\sin(ak_x) + i\sin(ak_y)\right) \\ -i\Delta^*\left(\sin(ak_x) - i\sin(ak_y)\right) & t\left(\cos(ak_x)+\cos(ak_y)\right) + (\frac{\mu}{2} - 2t) \end{pmatrix} \Psi_{\mathbf{k}} \tag{24}$$

where $\Psi^\dagger_{\mathbf{k}} = \begin{pmatrix} \mathbf{c}^\dagger_{k_x,k_y} & \mathbf{c}_{-k_x,-k_y} \end{pmatrix}$ and $\Psi_{\mathbf{k}} = \begin{pmatrix} \mathbf{c}_{k_x,k_y} \\ \mathbf{c}^\dagger_{-k_x,-k_y} \end{pmatrix}$ are the Nambu spinors. In the continuum limit ($\mathbf{k} \to 0$):

$$\mathcal{H} = \frac{1}{2}\sum_{k_x,k_y} \Psi^\dagger_{\mathbf{k}} \begin{pmatrix} \frac{\mathbf{k}^2}{2m} - \mu & 2i\Delta\left(k_x + ik_y\right) \\ -2i\Delta^*\left(k_x - ik_y\right) & -\frac{\mathbf{k}^2}{2m} + \mu \end{pmatrix} \Psi_{\mathbf{k}} \tag{25}$$

where $m = \frac{1}{2ta^2}$ and $a\Delta \to \Delta$. We see that the continuum limit has the characteristic $p_x + ip_y$ chiral form for the pairing potential.

2.2 Including vector potentials - Peierls substitution

We give a simple derivation for the Peierls substitution, which is based on Feynman's Lectures (Vol. III, Chapter 21). Our starting point is Hofstadter Hamiltonian:

$$H_0 = \sum_{m,n}\Big(-te^{i\theta^x_{m,n}}|m{+}1,n\rangle\langle m,n| - te^{i\theta^y_{m,n}}|m,n{+}1\rangle\langle m,n| - \epsilon_0|m,n\rangle\langle m,n|\Big) + \text{h.c.} \tag{26}$$

The translation operator $|m+1\rangle\langle m|$ can be written explicitly using its generator, that is the momentum operator. Under this representation its easy to expand it up to the second order,

$$|m{+}a\rangle\langle m| = \exp\left(-\frac{\mathbf{p}_x a}{\hbar}\right)|m\rangle\langle m| = \left(1 - \frac{i\mathbf{p}_x}{\hbar}a - \frac{\mathbf{p}_x^2}{2\hbar^2}a^2 + O(a^3)\right)|m\rangle\langle m| \tag{27}$$

and in a 2D lattice $|m{+}a\rangle\langle m| \longrightarrow |m,n{+}a\rangle\langle m,n|$. Next, we expand up to the second order the phase factors,

$$e^{i\theta} = 1 - i\theta' a - \frac{1}{2}\left(\theta'^2 + i\theta''^2\right)a^2 = 1 + \frac{ieA_x}{\hbar}a - \frac{e^2A_x^2}{2\hbar^2}a^2 + \frac{ieA'_x}{2\hbar}a^2 + O(a^3) \tag{28}$$

where for brevity with denoted: $\theta = \theta^x_{m,n}$, $A_x = \theta' = \partial_a\theta^x_{m,n}\big|_{a=0}$ and $A'_x = \theta'' = \partial_a^2\theta^x_{m,n}\big|_{a=0}$ with $\hbar = e = 1$. Substituting these expansions to relevant part of the Hamiltonian yields

$$\begin{aligned} &e^{i\theta}|m+a\rangle\langle m| + e^{-i\theta}|m\rangle\langle m+a| = \\ &\left(1 + \frac{ieA_x}{\hbar}a - \frac{e^2A_x^2}{2\hbar^2}a^2 + \frac{ieA'_x}{2\hbar}a^2 + O(a^3)\right)\left(1 - \frac{i\mathbf{p}_x}{\hbar}a - \frac{\mathbf{p}_x^2}{2\hbar^2}a^2 + O(a^3)\right)|m\rangle\langle m| + \text{h.c} = \\ &\left(2 - \frac{\mathbf{p}_x^2}{\hbar^2}a^2 + \frac{e\{\mathbf{p}_x, A_x\}}{\hbar^2}a^2 - \frac{e^2A_x^2}{\hbar^2}a^2 + O(a^3)\right)|m\rangle\langle m| \\ &\approx \left(-\frac{a^2}{\hbar^2}\left(\mathbf{p}_x - eA_x\right)^2 + 2 + O(a^3)\right)|m\rangle\langle m| \end{aligned} \tag{29}$$

Generalizing the last result to the 2D case, the we arrive to the Hamiltonian of a 2D electron gas at the continuum limit:

$$H_0 = \frac{1}{2m}\left(\mathbf{p} - e\mathbf{A}\right)^2 + \tilde{\epsilon}_0 \tag{30}$$

where the effective mass is $m = \hbar^2/2ta^2$ and $\tilde{\epsilon}_0 = \epsilon_0 + 4$.

Moreover, another useful result is

$$\lambda\left(e^{-i\theta}|m,\uparrow\rangle\langle m+a,\downarrow| - e^{i\theta}|m+a,\uparrow\rangle\langle m,\downarrow|\right) \approx \frac{2ia\lambda}{\hbar}|m,\uparrow\rangle(\mathbf{p}_x - A_x)\langle m,\downarrow| + O(a^3) \tag{31}$$

which is relevant when considering the Rashba Hamiltonian, $H_R = \alpha(\boldsymbol{\sigma}\times\mathbf{p})\cdot\hat{z}$ where $\alpha = 2a\lambda/\hbar$ is the Rashba coupling , $\boldsymbol{p}$ is the momentum and $\boldsymbol{\sigma}$ is the Pauli matrix vector. This is nothing but a two-dimensional version of the Dirac Hamiltonian (with a 90 degrees rotation of the spins).

2.3 Including vortex defects - exploiting the gauge invariant property

In this section we show how the phase factor of the coupling terms, which comprise all the information about the vortex defects change as we transform from the continuum limit to a discrete model. Staring from the p-wave continuum Hamiltonian

$$\mathcal{H} = \frac{1}{2}\begin{pmatrix}\psi^\dagger_{\mathbf{x},t} & \psi_{\mathbf{x},t}\end{pmatrix}\begin{pmatrix}\frac{1}{2m}(-\mathbf{p}+\mathbf{A})^2 - \mu & \left\{\Delta, \mathbf{p}_x - i\mathbf{p}_y\right\} \\ \left\{\overline{\Delta}, \mathbf{p}_x + i\mathbf{p}_y\right\} & -\frac{1}{2m}(\mathbf{p}+\mathbf{A})^2 + \mu\end{pmatrix}\begin{pmatrix}\psi_{\mathbf{x},t} \\ \psi^\dagger_{\mathbf{x},t}\end{pmatrix} \tag{32}$$

with $\Delta = {}^{\Delta_0}/_2\, e^{i\phi(\mathbf{x})}$ and $\phi(\mathbf{x}) = \sum_i \mathrm{Arg}(\mathbf{x} - \mathbf{x}_i)$, we apply a gauge transformation $\mathbf{U} = e^{i\tau_z\phi/2}$. The gauge transmutes the phase factor of the order parameter into fictional vector potential, $\boldsymbol{\nabla}\phi(\mathbf{x})$ and the Hamiltonian takes the following form

$$\mathcal{H} = \frac{1}{2}\begin{pmatrix}\psi^\dagger_{\mathbf{x},t}e^{i\phi(\mathbf{x})/2} & \psi_{\mathbf{x},t}e^{-i\phi(\mathbf{x})/2}\end{pmatrix}\begin{pmatrix}\frac{1}{2m}(-\mathbf{p}+\mathbf{a})^2 - \mu & \Delta_0(\mathbf{p}_x - i\mathbf{p}_y) \\ \Delta_0(\mathbf{p}_x + i\mathbf{p}_y) & -\frac{1}{2m}(\mathbf{p}+\mathbf{a})^2 + \mu\end{pmatrix}\begin{pmatrix}\psi_{\mathbf{x},t}e^{-i\phi(\mathbf{x})/2} \\ \psi^\dagger_{\mathbf{x},t}e^{i\phi(\mathbf{x})/2}\end{pmatrix} \tag{33}$$

with $\mathbf{a} = \mathbf{A} - \boldsymbol{\nabla}\phi(\mathbf{x})/2$. When discretizing the Hamiltonian there is a known prescription to deal with the vector potential - that is Peiers Substitution. Our last step is to use the $U(1)$ gauge transformation to recover the phase of the coupling terms. Under the transformation $\psi_{\mathbf{x}} \longrightarrow \psi_{\mathbf{x}}e^{i\phi(\mathbf{x})/2}$ the translation operator transforms as

$$\psi^\dagger_{\mathbf{x}+\boldsymbol{\delta}}\psi_{\mathbf{x}}\exp\left[-\frac{1}{2}\int_{\mathbf{x}}^{\mathbf{x}+\boldsymbol{\delta}} \mathrm{d}\mathbf{r}\boldsymbol{\nabla}\phi(\mathbf{r})\right] = \psi^\dagger_{\mathbf{x}+\boldsymbol{\delta}}\psi_{\mathbf{x}}\exp\left[\frac{1}{2}\Big(\phi(\mathbf{x}) - \phi(\mathbf{x}+\boldsymbol{\delta})\Big)\right] \rightarrow \psi^\dagger_{\mathbf{x}+\boldsymbol{\delta}}\psi_{\mathbf{x}} \tag{34}$$

while the coupling term transforms as

$$\psi^\dagger_{\mathbf{x}+\boldsymbol{\delta}}\psi^\dagger_{\mathbf{x}} \rightarrow \psi^\dagger_{\mathbf{x}+\boldsymbol{\delta}}\psi^\dagger_{\mathbf{x}}\exp\left[\frac{1}{2}\Big(\phi(\mathbf{x}) + \phi(\mathbf{x}+\boldsymbol{\delta})\Big)\right]. \tag{35}$$

Thus, we have found how to incorporate vortex defects into the tight-binding model.

2.4 Transforming the creation-annihilation operators from the lattice site occupation basis to the energy occupation basis

The lattice Hamiltonian is

$$\mathcal{H} = \sum_{m,n} \left[-t\left(\psi^\dagger_{m+1,n}\psi_{m,n} + \psi^\dagger_{m,n+1}\psi_{m,n}\right) - \frac{1}{2}(\mu - 4t)\psi^\dagger_{m,n}\psi_{m,n} \right.$$
$$\left. + \left(\Delta^{m+1,n}_{m,n}\psi^\dagger_{m+1,n}\psi^\dagger_{m,n} + i\Delta^{m,n+1}_{m,n}\psi^\dagger_{m,n+1}\psi^\dagger_{m,n}\right)\right] + \text{h.c.} \quad (36)$$

where the coupling constant in the presence of vortex defects is defined in Eq.(35). We rewrite it in the Nambu space formalism, $\mathcal{H} = \frac{1}{2}\sum_{m,n}\left[\Psi^\dagger H \Psi - (\mu - 4t)\right]$, where the Nambu spinor structure for a 2×2 lattice is simply $\Psi = (\psi_{11}, \psi_{12}, \psi_{21}, \psi_{22}, \psi^\dagger_{11}, \psi^\dagger_{12}, \psi^\dagger_{21}, \psi^\dagger_{22})^T$ and H is the corresponding Hamiltonian density matrix (also refereed as the single particle Hamiltonian and BdG (Bogoliubov-de-Gennes) Hamiltonian). This representation illuminates the particle-hole symmetry (PHS), which takes the form $\tau_x H^* \tau_x = -H$ with $\tau_x = \tau_1 \otimes I$ and I is a unity matrix with a dimension that equals to the lattice size. Consequently, the Nambu spinor can be divided into two vectors, $\Psi = (\psi, \psi^\dagger)^T$ and the BdG Hamiltonian can be divided into four matrix blocks,

$$H = \begin{pmatrix} \xi & \Delta \\ \Delta^\dagger & -\xi^T \end{pmatrix}. \quad (37)$$

Next, we diagonalize the single particle Hamiltonian in order to find a transformation of the creation-annihilation operators from the lattice site occupation basis to the energy occupation basis. A consequence of the PHS is that for every eigenenergy ϵ with eigenstate Ψ_ϵ there exist a opposite eigenenergy, $-\epsilon$ and an eigenstate $\tau_x \mathbf{c}^*_\epsilon$ -

$$\begin{array}{l} (a) \qquad \tau_x H^* \tau_x = -H \\ (b)\ \ H\begin{pmatrix} u_\epsilon & v_\epsilon \end{pmatrix}^T = \epsilon \begin{pmatrix} u_\epsilon & v_\epsilon \end{pmatrix}^T \end{array} \Rightarrow \tau_x H^* \tau_x \begin{pmatrix} u_\epsilon \\ v_\epsilon \end{pmatrix} = -H \begin{pmatrix} u_\epsilon \\ v_\epsilon \end{pmatrix} \Rightarrow H \begin{pmatrix} v_\epsilon \\ u_\epsilon \end{pmatrix}^* = -\epsilon \begin{pmatrix} v_\epsilon \\ u_\epsilon \end{pmatrix}^* \quad (38)$$

where u_ϵ and v_ϵ are both vectors of dimension that equals to the lattice size. Hence, we found a relation between the eigenstates with opposite eigenvalues, $u_{-\epsilon} = \bar{v}_\epsilon$. Since the BdG Hamiltonian is redundant the number of free non-interacting fermions is half of the Hamiltonian dimensionality. This can be seen by writing explicitly the basis transformation of the creation-annihilation operators -

$$\mathbf{c}^\dagger_\epsilon = \sum_{m,n}\left(u^{m,n}_\epsilon \psi^\dagger_{m,n} + v^{m,n}_\epsilon \psi_{m,n}\right), \qquad \mathbf{c}_\epsilon = \sum_{m,n}\left(\bar{u}^{m,n}_\epsilon \psi_{m,n} + \bar{v}^{m,n}_\epsilon \psi^\dagger_{m,n}\right) \quad (39)$$

but, as shown above, the PHS imply that

$$\mathbf{c}^\dagger_{-\epsilon} = \sum_{m,n}\left(\bar{v}^{m,n}_\epsilon \psi^\dagger_{m,n} + \bar{u}^{m,n}_\epsilon \psi_{m,n}\right) \quad (40)$$

which leads to a relationship between the creation-annihilation operators, i.e. $\mathbf{c}_\epsilon^\dagger = \mathbf{c}_{-\epsilon}$.

Now up to a constant the Hamiltonian after the diagonalization process is

$$\mathcal{H} \sim \frac{1}{2}\sum_{\epsilon} \epsilon\, \mathbf{c}_\epsilon^\dagger \mathbf{c}_\epsilon = \frac{1}{2}\sum_{\epsilon>0}\left(\epsilon\, \mathbf{c}_\epsilon^\dagger \mathbf{c}_\epsilon - \epsilon\, \mathbf{c}_{-\epsilon}^\dagger \mathbf{c}_{-\epsilon}\right) = \frac{1}{2}\sum_{\epsilon>0}\left(\epsilon\, \mathbf{c}_\epsilon^\dagger \mathbf{c}_\epsilon - \epsilon\, \mathbf{c}_\epsilon \mathbf{c}_\epsilon^\dagger\right) = \sum_{\epsilon>0} \epsilon\left(\mathbf{c}_\epsilon^\dagger \mathbf{c}_\epsilon - \frac{1}{2}\right) \tag{41}$$

The missing constant is just the ground state energy, i.e. $H = \sum_{\epsilon>0} \epsilon \left(\mathbf{c}_\epsilon^\dagger \mathbf{c}_\epsilon\right) + E_0$. The cause to write the Hamiltonian using only non-negative eigenvalues is that they are an artefact of the redundant formalism. For completeness of the discussion, we also give the inverse transformation

$$\psi_{m,n} = \sum_{\epsilon} u_{m,n}^{\epsilon} \mathbf{c}_\epsilon, \qquad \psi_{m,n}^\dagger = \sum_{\epsilon} v_{m,n}^{\epsilon} \mathbf{c}_\epsilon. \tag{42}$$

2.5 The anti-commutation relations of the excitation operators

$$\begin{aligned}
\left\{\mathbf{c}_\epsilon^\dagger \mathbf{c}_{\epsilon'}^\dagger\right\} = \sum_{\substack{m,n,\\ \tilde{m},\tilde{n}}} \Big[& u_\epsilon^{m,n} u_{\epsilon'}^{\tilde{m},\tilde{n}} \psi_{m,n}^\dagger \psi_{\tilde{m},\tilde{n}}^\dagger + v_\epsilon^{m,n} v_{\epsilon'}^{\tilde{m},\tilde{n}} \psi_{m,n}\psi_{\tilde{m},\tilde{n}} + u_\epsilon^{m,n} v_{\epsilon'}^{\tilde{m},\tilde{n}} \psi_{m,n}^\dagger \psi_{\tilde{m},\tilde{n}} + v_\epsilon^{m,n} u_{\epsilon'}^{\tilde{m},\tilde{n}} \psi_{m,n} \psi_{\tilde{m},\tilde{n}}^\dagger + \\
& u_{\epsilon'}^{\tilde{m},\tilde{n}} u_\epsilon^{m,n} \psi_{\tilde{m},\tilde{n}}^\dagger \psi_{m,n}^\dagger + v_{\epsilon'}^{\tilde{m},\tilde{n}} v_\epsilon^{m,n} \psi_{\tilde{m},\tilde{n}} \psi_{m,n} + u_{\epsilon'}^{\tilde{m},\tilde{n}} v_\epsilon^{m,n} \psi_{\tilde{m},\tilde{n}}^\dagger \psi_{m,n} + v_{\epsilon'}^{\tilde{m},\tilde{n}} u_\epsilon^{m,n} \psi_{\tilde{m},\tilde{n}} \psi_{m,n}^\dagger + \Big] = \\
\sum_{\substack{m,n\\ \tilde{m},\tilde{n}}} \Big[& u_\epsilon^{m,n} u_{\epsilon'}^{\tilde{m},\tilde{n}} \left\{\psi_{m,n}^\dagger, \psi_{\tilde{m},\tilde{n}}^\dagger\right\} + v_\epsilon^{m,n} v_{\epsilon'}^{\tilde{m},\tilde{n}} \left\{\psi_{m,n}, \psi_{\tilde{m},\tilde{n}}\right\} + u_\epsilon^{m,n} v_{\epsilon'}^{\tilde{m},\tilde{n}} \left\{\psi_{m,n}^\dagger, \psi_{\tilde{m},\tilde{n}}\right\} + \\
& v_\epsilon^{m,n} u_{\epsilon'}^{\tilde{m},\tilde{n}} \left\{\psi_{m,n}, \psi_{\tilde{m},\tilde{n}}^\dagger\right\}\Big] = \sum_{\substack{m,n\\ \tilde{m},\tilde{n}}} \left[u_\epsilon^{m,n} v_{\epsilon'}^{\tilde{m},\tilde{n}} \delta_{m,n}\delta_{\tilde{m},\tilde{n}} + v_\epsilon^{m,n} u_{\epsilon'}^{\tilde{m},\tilde{n}} \delta_{m,n}\delta_{\tilde{m},\tilde{n}}\right] = \\
& \sum_{m,n} \left[u_\epsilon^{m,n} v_{\epsilon'}^{m,n} + v_\epsilon^{m,n} u_{\epsilon'}^{m,n}\right] = \langle \varphi_\epsilon | \boldsymbol{\tau}_x \mathbf{K} | \varphi_{\epsilon'}\rangle = \langle \varphi_\epsilon | \varphi_{-\epsilon'}\rangle = 0
\end{aligned} \tag{43}$$

In a similar fashion we'll get that

$$\{\mathbf{c}_\epsilon \mathbf{c}_{\epsilon'}\} = \langle \varphi_\epsilon | \boldsymbol{\tau}_x \mathbf{K} | \varphi_{\epsilon'}\rangle^* = 0 \tag{44}$$

and also

$$\left\{\mathbf{c}_\epsilon \mathbf{c}^\dagger_{\epsilon'}\right\} = \sum_{\substack{m,n,\\ \tilde{m},\tilde{n}}} \Big[\bar{u}^{m,n}_\epsilon u^{\tilde{m},\tilde{n}}_{\epsilon'} \psi_{m,n}\psi^\dagger_{\tilde{m},\tilde{n}} + \bar{v}^{m,n}_\epsilon v^{\tilde{m},\tilde{n}}_{\epsilon'} \psi^\dagger_{m,n}\psi_{\tilde{m},\tilde{n}} + \bar{u}^{m,n}_\epsilon v^{\tilde{m},\tilde{n}}_{\epsilon'} \psi_{m,n}\psi_{\tilde{m},\tilde{n}} + \bar{v}^{m,n}_\epsilon u^{\tilde{m},\tilde{n}}_{\epsilon'} \psi^\dagger_{m,n}\psi^\dagger_{\tilde{m},\tilde{n}} +$$

$$u^{\tilde{m},\tilde{n}}_{\epsilon'} \bar{u}^{m,n}_\epsilon \psi^\dagger_{\tilde{m},\tilde{n}}\psi_{m,n} + v^{\tilde{m},\tilde{n}}_{\epsilon'} \bar{v}^{m,n}_\epsilon \psi_{\tilde{m},\tilde{n}}\psi^\dagger_{m,n} + u^{\tilde{m},\tilde{n}}_{\epsilon'} \bar{v}^{m,n}_\epsilon \psi^\dagger_{\tilde{m},\tilde{n}}\psi^\dagger_{m,n} + v^{\tilde{m},\tilde{n}}_{\epsilon'} \bar{u}^{m,n}_\epsilon \psi_{\tilde{m},\tilde{n}}\psi_{m,n}\Big] =$$

$$\sum_{\substack{m,n,\\ \tilde{m},\tilde{n}}} \Big[\bar{u}^{m,n}_\epsilon u^{\tilde{m},\tilde{n}}_{\epsilon'} \left\{\psi_{m,n}, \psi^\dagger_{\tilde{m},\tilde{n}}\right\} + \bar{v}^{m,n}_\epsilon v^{\tilde{m},\tilde{n}}_{\epsilon'} \left\{\psi^\dagger_{m,n}, \psi_{\tilde{m},\tilde{n}}\right\} + \bar{u}^{m,n}_\epsilon v^{\tilde{m},\tilde{n}}_{\epsilon'} \left\{\psi_{m,n}, \psi_{\tilde{m},\tilde{n}}\right\} +$$

$$\bar{v}^{m,n}_\epsilon u^{\tilde{m},\tilde{n}}_{\epsilon'} \left\{\psi^\dagger_{m,n}, \psi^\dagger_{\tilde{m},\tilde{n}}\right\}\Big] = \sum_{\substack{m,n\\ \tilde{m},\tilde{n}}} \left[\bar{u}^{m,n}_\epsilon u^{\tilde{m},\tilde{n}}_{\epsilon'} \delta_{m,n}\delta_{\tilde{m},\tilde{n}} + \bar{v}^{m,n}_\epsilon v^{\tilde{m},\tilde{n}}_{\epsilon'} \delta_{m,n}\delta_{\tilde{m},\tilde{n}}\right] =$$

$$\sum_{m,n} \left[\bar{u}^{m,n}_\epsilon u^{m,n}_{\epsilon'} + \bar{v}^{m,n}_{\epsilon'} v^{m,n}_\epsilon\right] = \langle\varphi_\epsilon|\varphi_{\epsilon'}\rangle = \delta_{\epsilon,\epsilon'}. \quad (45)$$

Since we are dealing only with non-negative eigenenergies, the last equality always holds. A special attention should be taken in the case that both energies are zero , i.e. $\epsilon = \epsilon'$. Since there is a degeneracy, one must choose such eigenstates that obey the PHS, i.e. $\varphi_{0^+} = \tau_x \bar{\varphi}_{0^-}$.

2.6 Derivation of the p-wave superconductor many body ground state

Using the annihilation operator in the energy basis,

$$\mathbf{c}_\epsilon = \sum_{m,n} \left(\bar{v}^{m,n}_\epsilon \psi^\dagger_{m,n} + \bar{u}^{m,n}_\epsilon \psi_{m,n}\right) \quad (46)$$

we write the unnormalized many-body ground state as

$$|gs\rangle \propto \prod_{0\le\epsilon} \mathbf{c}_\epsilon|0\rangle = \prod_{0\le\epsilon}\sum_{m,n} \left(\bar{v}^{m,n}_\epsilon \psi^\dagger_{m,n} + \bar{u}^{m,n}_\epsilon \psi_{m,n}\right)|0\rangle. \quad (47)$$

which satisfies $\mathbf{c}_{\epsilon'}|gs\rangle =$ for any $\epsilon' > 0$.

2.7 The Streda Formula

The expression for the total particle density is

$$\rho = \frac{\delta S}{\delta a_0} = C_1 a_0 - \sigma_{xy}(\nabla\times\mathbf{a})_z + C_0 \quad (48)$$

where C_1, C_2 and σ_{xy} are constants that depend on the structure of the system, i.e. continuum model, square lattice, etc. Since a_0 is coupled to τ_3, the local particle density is given by

$$\rho_{\mathbf{x},t} = \frac{1}{2}\left\langle \Psi^\dagger_{\mathbf{x},t}\tau_3\Psi_{\mathbf{x},t}\right\rangle = \frac{1}{2}\left\langle \psi^\dagger_{\mathbf{x},t}\psi_{\mathbf{x},t} - \psi_{\mathbf{x},t}\psi^\dagger_{\mathbf{x},t}\right\rangle = \left\langle \psi^\dagger_{\mathbf{x},t}\psi_{\mathbf{x},t}\right\rangle - \frac{1}{2} \quad (49)$$

Using Eq.(48-49) we can write a simple expression of the Hall like conductivity (also known as the Streda formula), σ_{xy} for the lattice model,

$$\sigma_{xy} = \frac{\partial \rho}{\partial B_z} = \sum_{m,n} \frac{\partial \left\langle \psi^\dagger_{m,n} \psi_{m,n} \right\rangle}{\partial \phi} \tag{50}$$

where the ϕ is the total flux through the whole lattice and the ground state exception value of the local particle number is:

$$\begin{aligned}
\left\langle \psi^\dagger_{m,n} \psi_{m,n} \right\rangle &= \sum_{\epsilon_1, \epsilon_2} v^{\epsilon_1}_{m,n} u^{\epsilon_2}_{m,n} \left\langle c_{\epsilon_1} c_{\epsilon_2} \right\rangle = \left(\sum_{\substack{0^+ \leqslant \epsilon_1 \\ 0^+ \leqslant \epsilon_2}} + \sum_{\substack{0^+ \leqslant \epsilon_1 \\ \epsilon_2 \leqslant 0^-}} + \sum_{\substack{\epsilon_1 \leqslant 0^- \\ 0^+ \leqslant \epsilon_2}} + \sum_{\substack{\epsilon_1 \leqslant 0^- \\ \epsilon_2 \leqslant 0^-}} \right) v^{\epsilon_1}_{m,n} u^{\epsilon_2}_{m,n} \left\langle c_{\epsilon_1} c_{\epsilon_2} \right\rangle = \\
&= \sum_{\substack{0^+ \leqslant \epsilon_1 \\ 0^+ \leqslant \epsilon_2}} \left(v^{\epsilon_1}_{m,n} u^{\epsilon_2}_{m,n} \left\langle c_{\epsilon_1} c_{\epsilon_2} \right\rangle + v^{\epsilon_1}_{m,n} \bar{v}^{\epsilon_2}_{m,n} \left\langle c_{\epsilon_1} c^\dagger_{\epsilon_2} \right\rangle + \bar{u}^{\epsilon_1}_{m,n} u^{\epsilon_2}_{m,n} \left\langle c^\dagger_{\epsilon_1} c_{\epsilon_2} \right\rangle + \bar{u}^{\epsilon_1}_{m,n} \bar{v}^{\epsilon_1}_{m,n} \left\langle c^\dagger_{\epsilon_1} c^\dagger_{\epsilon_2} \right\rangle \right) \\
&= \begin{cases} \sum_{0<\epsilon} |v^{\epsilon}_{m,n}|^2 + |v^{0^+}_{m,n}|^2 & \text{, for an even ground state} \\ \sum_{0<\epsilon} |v^{\epsilon}_{m,n}|^2 + |u^{0^+}_{m,n}|^2 & \text{, for an odd ground state} \end{cases},
\end{aligned} \tag{51}$$

where we used the operator transformation appearing in Eq.(42)and the PHS relations appearing in Eq.(38) and Eq.(40). Also, we note that $|v^{0^+}_{m,n}| = |u^{0^+}_{m,n}|$ whenever $\epsilon^{0+} = \epsilon^{0-}$.[1]

2.8 Supercurrents

We aim to calculate the supercurrents induced in the system as the magnetic field penetrates it [3, 1]. Our starting point is the continuity equation

$$\partial_t \boldsymbol{\rho}_{ij} + (\boldsymbol{\nabla} \cdot \mathbf{j})_{ij} = 0 \tag{52}$$

where $\boldsymbol{\rho}_{ij} = e\mathbf{n}_{ij}$ and $\mathbf{j}$ are the electron density and current operators respectively. On the other side, In the Heisenberg picture the equation of motion for the particle density operator at site i, j is given by

$$\dot{\mathbf{n}}_{ij} = \frac{i}{\hbar} \left[\mathcal{H}, \mathbf{n}_{ij} \right]. \tag{53}$$

Thus, we identify the divergence of the current as

$$(\boldsymbol{\nabla} \cdot \hat{\mathbf{j}})_{ij} = -\frac{ie}{\hbar} \left[\mathcal{H}, \mathbf{n}_{ij} \right]. \tag{54}$$

where both the kinetic and pairing terms in the BCS Hamiltonian do not commute with the particle number operator. The total (sum for all sites) particle number operator is expected to commute with the Hamiltonian in an isolated system,

[1] $\langle gs|\psi^\dagger_{m,n}\psi_{m,n}|gs\rangle$ is the groundstate expectation value of the electrons number. In addition, the occupation number of the Bogoliubov quasiparticles is $\langle gs|\mathbf{c}_\epsilon \mathbf{c}^\dagger_\epsilon|gs\rangle = 1$ for $\epsilon \geq 0^+$ and 0 otherwise.

otherwise source terms must be included in the continuity equation. However, one of the mean-field theory anomalies is the absent of particle number conservation, so the pairing terms do not commute with the total particle number. This anomaly leads to a contradiction since the total net-current is expected to be zero in an isolated superconductor. The problem is overcome by using instead the exact many-body interaction term. For a p-wave superconductor we have spinless fermions and interaction term takes the form

$$\mathcal{H}_{int} = \sum_{ij} V_{ij} \psi_i^\dagger \psi_j^\dagger \psi_j \psi_i. \tag{55}$$

As shown in the calculation below, $[\mathcal{H}_{int}, \mathbf{n}_{ij}] = 0$, namely the number operator and the exact interaction term commute.

$$\psi_i \psi_k^\dagger \psi_k = (\delta_{ik} - \psi_k^\dagger \psi_i)\psi_k = \delta_{ik}\psi_i + \psi_k^\dagger \psi_k \psi_i \quad \Rightarrow \quad [\psi_i, \psi_k^\dagger \psi_k] = \psi_k \delta_{ik} \tag{56}$$

$$\left(\left[\psi_i, \psi_k^\dagger \psi_k\right]\right)^\dagger = \psi_k^\dagger \psi_k \psi_i^\dagger - \psi_i^\dagger \psi_k^\dagger \psi_k = -\left[\psi_i^\dagger, \psi_k^\dagger \psi_k\right] \quad \Rightarrow \quad [\psi_i^\dagger, \psi_k^\dagger \psi_k] = -\psi_k^\dagger \delta_{ik} \tag{57}$$

$$\begin{aligned}\psi_i^\dagger \psi_j^\dagger \psi_j \psi_i \mathbf{n}_k = \psi_i^\dagger \psi_j^\dagger \psi_j \mathbf{n}_k \psi_i + \psi_k^\dagger \psi_j^\dagger \psi_j \psi_k = \psi_i^\dagger \psi_j^\dagger \mathbf{n}_k \psi_j \psi_i + \psi_i^\dagger \psi_k^\dagger \psi_k \psi_i + \psi_k^\dagger \psi_j^\dagger \psi_j \psi_k = \\ \psi_i^\dagger \mathbf{n}_k \psi_j^\dagger \psi_j \psi_i + \psi_k^\dagger \psi_j^\dagger \psi_j \psi_k = \mathbf{n}_k \psi_i^\dagger \psi_j^\dagger \psi_j \psi_i \quad \Rightarrow \quad \left[\psi_i^\dagger \psi_j^\dagger \psi_j \psi_i, \mathbf{n}_k\right] = 0\end{aligned} \tag{58}$$

Hence, only the kinetic part of the BCS Hamiltonian play a a role in the divergence of the current operator,

$$(\boldsymbol{\nabla} \cdot \hat{\mathbf{j}})_{ij} = -\frac{ie}{\hbar} \left[\mathcal{H}_0, \mathbf{n}_{ij}\right]. \tag{59}$$

The mean-field approximation would take a role only when calculating the exception value of the operator.

Our next step is to calculate the current operator for a p-wave superconductor on a lattice.

$$\begin{aligned}t_{ij}\psi_i^\dagger \psi_j \mathbf{n}_k = t_{ij}\psi_i^\dagger \mathbf{n}_k \psi_j + t_{ik}\psi_i^\dagger \psi_k = t_{ij}\mathbf{n}_k \psi_i^\dagger \psi_j - t_{kj}\psi_k^\dagger \psi_j + t_{ik}\psi_i^\dagger \psi_k \\ \Rightarrow \quad \left[t_{ij}\psi_i^\dagger \psi_j, \mathbf{n}_k\right] = t_{ik}\psi_i^\dagger \psi_k - t_{kj}\psi_k^\dagger \psi_j\end{aligned} \tag{60}$$

For a square lattice $\mathcal{H}_0$, the kinetic part is

$$\mathcal{H}_0 = \sum_{a,b,c,d} \left[r_{a,b}^{c,d} \psi_{a,b}^\dagger \psi_{c,d} + u_{a,b}^{c,d} \psi_{a,b}^\dagger \psi_{c,d} + \epsilon_{c,d}^{a,b} \psi_{a,b}^\dagger \psi_{c,d} + \text{h.c} \right] \tag{61}$$

where the right (up) hopping amplitude is $r_{c,d}^{a,b} \propto \delta_{a,c-1}\delta_{b,d}$ ($u_{c,d}^{a,b} \propto \delta_{a,c}\delta_{b+1,d}$) and the on-site energy is $\epsilon_{c,d}^{a,b} \propto \delta_{a,c}\delta_{b,d}$. The commutation relation between the $\mathcal{H}_0$ and $\mathbf{n}_{i,j}$ is

$$\begin{aligned}\left[\mathcal{H}_0, \mathbf{n}_{i,j}\right] = \sum_{a,b} \Big[& r_{i,j}^{a,b} \psi_{a,b}^\dagger \psi_{i,j} - r_{a,b}^{i,j} \psi_{i,j}^\dagger \psi_{a,b} + u_{i,j}^{a,b} \psi_{a,b}^\dagger \psi_{i,j} - u_{a,b}^{i,j} \psi_{i,j}^\dagger \psi_{a,b} \\ & + \epsilon_{i,j}^{a,b} \psi_{a,b}^\dagger \psi_{i,j} - \epsilon_{a,b}^{i,j} \psi_{i,j}^\dagger \psi_{a,b} - \text{h.c} \Big].\end{aligned} \tag{62}$$

where we subtracted the hermitian conjugate (instead of adding as in the Hamiltonian) since

$$\Big(\big[t_{ij}\psi_i^\dagger\psi_j,\mathbf{n}_k\big]\Big)^\dagger = \big(t_{ij}\psi_i^\dagger\psi_j\mathbf{n}_k - t_{ij}\mathbf{n}_k\psi_i^\dagger\psi_j\big)^\dagger = \bar{t}_{ij}\mathbf{n}_k\psi_j^\dagger\psi_i - \bar{t}_{ij}\psi_j^\dagger\psi_i\mathbf{n}_k = -\big[\bar{t}_{ij}\psi_j^\dagger\psi_i,\mathbf{n}_k\big]$$
$$\Rightarrow \quad \big[(t_{ij}\psi_i^\dagger\psi_j)^\dagger,\mathbf{n}_k\big] = -\Big(\big[t_{ij}\psi_i^\dagger\psi_j,\mathbf{n}_k\big]\Big)^\dagger. \quad (63)$$

Hence, the divergence of the current operator is

$$(\nabla \cdot \hat{\mathbf{j}})_{ij} = -\frac{ie}{\hbar}\Big(t_{i,j}^{i+1,j}\psi_{i+1,j}^\dagger\psi_{i,j} - t_{i-1,j}^{i,j}\psi_{i,j}^\dagger\psi_{i-1,j} + t_{i,j}^{i,j+1}\psi_{i,j+1}^\dagger\psi_{i,j} - t_{i,j-1}^{i,j}\psi_{i,j}^\dagger\psi_{i,j-1} - \text{h.c}\Big) \quad (64)$$

where $t_{k,l}^{i,j}$ is the hopping amplitude[2]. This equation tells us that the divergence of the current at site i, j is equal to the current going out minus the current going in. Thus, the net current through all the nodes is conserved. We processed by taking the groundstate exception value of the current operator divergence in the same fashion as in Eq.(51),

$$\langle\nabla \cdot \hat{\mathbf{j}}\rangle_{ij} = -\frac{ie}{\hbar}\sum_{0^+\leq\epsilon}\Big(t_{i,j}^{i+1,j}v_{i+1,j}^\epsilon\bar{v}_{i,j}^\epsilon - t_{i-1,j}^{i,j}v_{i,j}^\epsilon\bar{v}_{i-1,j}^\epsilon + t_{i,j}^{i,j+1}v_{i,j+1}^\epsilon\bar{v}_{i,j}^\epsilon - t_{i,j-1}^{i,j}v_{i,j}^\epsilon\bar{v}_{i,j-1}^\epsilon\Big) + \text{c.c} \quad (65)$$

Calculating the average of currents going in and out a node would give a good estimation of the current density at the node.

2.9 Transforming from a p-wave superconductor tight-binding model of a triangular lattice to the continuum limit

We begin with the p-wave lattice Hamiltonian,

$$\mathcal{H} = \sum_{m,n}\Bigg[-t\Big(\psi_{m+2,n}^\dagger\psi_{m,n} + \psi_{m+1,n+1}^\dagger\psi_{m,n} + \psi_{m-1,n+1}^\dagger\psi_{m,n}\Big) - \frac{1}{2}(\mu - 6t)\psi_{m,n}^\dagger\psi_{m,n}$$
$$+\Delta\Big(\psi_{m+2,n}^\dagger\psi_{m,n}^\dagger + e^{i\pi/3}\psi_{m+1,n+1}^\dagger\psi_{m,n}^\dagger + e^{i2\pi/3}\psi_{m-1,n+1}^\dagger\psi_{m,n}^\dagger\Big) + \text{h.c} \quad (66)$$

The sum $\sum_{m,n}$ corresponds to the $\sum_{k=0}^{q_x-1}\sum_{l=0}^{q_y-1}$ with the coordinates (m, n) related to (k, l) by $m = 2k + \text{mod}(n, 2)$ and $n = l$.

Our next step is to transform the Hamiltonian into momentum space. The creation and annihilation operators can be represented as a discrete Fourier transform

$$\psi_{m,n} = \frac{1}{\sqrt{N}}\sum_{k_x,k_y} e^{ik_x ma/2 + ik_y n\sqrt{3}a/2}\mathbf{c}_{k_x,k_y}, \quad (67)$$

where for a lattice with $q_x \times q_y$ atomic sites $N = q_x \times q_y$. In addition, due to the periodic boundary conditions $k_i q_i a_i = 2\pi p_i$, $p_i \in \mathbb{Z}$ and $i = x, y$.

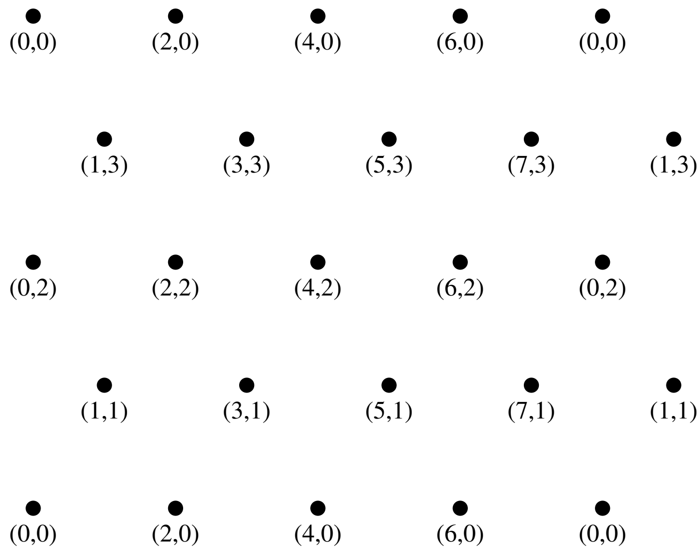

Figure 1: **Triangular lattice with periodic boundary conditions.** The indices that appear below the nodes correspond to the tight-binding Hamiltonian indices, (m, n). The lattice sites are positioned at $(ma/2, \sqrt{3}na/2)$ where a is the lattice constant.

The translation operators transform as

$$\begin{aligned}\sum_{m,n}\psi^\dagger_{m+2,n}\psi_{m,n} + \text{h.c.} &= 2\sum_{k_x,k_y}\cos(k_x a)\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{k_x,k_y} \\ \sum_{m,n}\psi^\dagger_{m+1,n+1}\psi_{m,n} + \text{h.c.} &= 2\sum_{k_x,k_y}\cos\left(\frac{k_x a}{2} + \frac{\sqrt{3}k_y a}{2}\right)\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{k_x,k_y} \\ \sum_{m,n}\psi^\dagger_{m-1,n+1}\psi_{m,n} + \text{h.c.} &= 2\sum_{k_x,k_y}\cos\left(-\frac{k_x a}{2} + \frac{\sqrt{3}k_y a}{2}\right)\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{k_x,k_y}\end{aligned} \tag{68}$$

The occupation number operator transforms as

$$\sum_{m,n}\psi^\dagger_{m,n}\psi_{m,n} = \sum_{k_x,k_y}\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}_{k_x,k_y} \tag{69}$$

so the kinetic part of the Hamiltonian is

$$\begin{aligned}-2t\left(\cos(k_x a) + \cos\left(\frac{k_x a}{2} + \frac{\sqrt{3}k_y a}{2}\right) + \cos\left(-\frac{k_x a}{2} + \frac{\sqrt{3}k_y a}{2}\right)\right) - (\mu - 6t) \approx \\ t\left((k_x a)^2 + \left(\frac{k_x a}{2} + \frac{\sqrt{3}k_y a}{2}\right)^2 + \left(-\frac{k_x a}{2} + \frac{\sqrt{3}k_y a}{2}\right)^2\right) - \mu = \\ \frac{3t}{2}\left((k_x a)^2 + (k_y a)^2\right) - \mu.\end{aligned} \tag{70}$$

The coupling terms transform as

$$\begin{aligned}\sum_{m,n}\psi^\dagger_{m+2,n}\psi^\dagger_{m,n} &= i\sum_{k_x,k_y}\sin(k_xa)\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}^\dagger_{-k_x,-k_y}\\ \sum_{m,n}\psi^\dagger_{m+1,n+1}\psi^\dagger_{m,n} &= i\sum_{k_x,k_y}\sin\left(\frac{k_xa}{2}+\frac{\sqrt{3}k_y}{2}\right)\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}^\dagger_{-k_x,-k_y}\\ \sum_{m,n}\psi^\dagger_{m+1,n-1}\psi^\dagger_{m,n} &= i\sum_{k_x,k_y}\sin\left(-\frac{k_xa}{2}+\frac{\sqrt{3}k_y}{2}\right)\mathbf{c}^\dagger_{k_x,k_y}\mathbf{c}^\dagger_{-k_x,-k_y}\end{aligned}\tag{71}$$

so the in interaction part of the Hamiltonian is

$$\begin{aligned}&i\Delta\left(\sin(k_xa)+e^{i\pi/3}\sin\left(\frac{k_xa}{2}+\frac{\sqrt{3}k_y}{2}\right)+e^{i2\pi/3}\sin\left(-\frac{k_xa}{2}+\frac{\sqrt{3}k_y}{2}\right)\right)\approx\\ &\quad i\Delta\left(k_xa+\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\left(\frac{k_xa}{2}+\frac{\sqrt{3}k_ya}{2}\right)+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\left(-\frac{k_xa}{2}+\frac{\sqrt{3}k_ya}{2}\right)\right)=\\ &\quad i\Delta\frac{3a}{2}(k_x+ik_y)\end{aligned}\tag{72}$$

The complete Hamiltonian density can be written as

$$\begin{aligned}H=&\left[-t\left(\cos(k_xa)+\cos\left(\frac{k_xa}{2}+\frac{\sqrt{3}k_ya}{2}\right)+\cos\left(-\frac{k_xa}{2}+\frac{\sqrt{3}k_ya}{2}\right)\right)-\frac{1}{2}(\mu-6t)\right]\tau_z\\ &-\frac{\Delta}{2}\left[2\sin(k_xa)+\sin\left(\frac{k_xa}{2}+\frac{\sqrt{3}k_ya}{2}\right)-\sin\left(-\frac{k_xa}{2}+\frac{\sqrt{3}k_ya}{2}\right)\right]\tau_y\\ &-\frac{\sqrt{3}\Delta}{2}\left[\sin\left(\frac{k_xa}{2}+\frac{\sqrt{3}k_ya}{2}\right)+\sin\left(-\frac{k_xa}{2}+\frac{\sqrt{3}k_ya}{2}\right)\right]\tau_x\end{aligned}\tag{73}$$

and the Hamiltonian is just $\mathcal{H}=\Psi^\dagger_{\mathbf{k}}H\Psi_{\mathbf{k}}$ with the Nambu spinors $\Psi^\dagger_{\mathbf{k}}=\begin{pmatrix}\mathbf{c}^\dagger_{k_x,k_y} & \mathbf{c}_{-k_x,-k_y}\end{pmatrix}$ and $\Psi_{\mathbf{k}}=\begin{pmatrix}\mathbf{c}_{k_x,k_y}\\ \mathbf{c}^\dagger_{-k_x,-k_y}\end{pmatrix}$.

In the continuum limit the Hamiltonian takes the standard form -

$$\mathcal{H}=\frac{1}{2}\sum_{k_x,k_y}\Psi^\dagger_{\mathbf{k}}\begin{pmatrix}\frac{\mathbf{k}^2}{2m}-\mu & \Delta\left(k_x+ik_y\right)\\ \Delta^*\left(k_x-ik_y\right) & -\frac{\mathbf{k}^2}{2m}+\mu\end{pmatrix}\Psi_{\mathbf{k}}\tag{74}$$

where the effective mass is $m=\frac{1}{3ta^2}$ and $i3a\Delta\rightarrow\Delta$.

2.10 Fermion parity switches

To understand the existence of crossing between two energy levels at zero energy appear, recall that to obtain a Bogoliubov-de Gennes description of the superconductor we had to double the number of degrees of freedom by introducing holes. Hence, a pair of $\pm E$ energy levels does not corresponds to two distinct quantum states, but to a single quantum state: $\mathbf{c}_\epsilon^\dagger = \mathbf{c}_{-\epsilon}$. Since $|gs\rangle \propto \prod_{0\leq\epsilon} \mathbf{c}_\epsilon|0\rangle$, populating the partner state at energy E is the same as emptying the positive energy state. In other words, at each crossing the fermion parity in the ground state of the superconductor changes from even to odd, or vice versa. Hence these crossings are fermion parity switches.

Since the ground state fermion parity is preserved by the superconducting Hamiltonian if there are no Bogoliubov quasiparticles crossing zero energy, the ground state fermion parity is the topological invariant of this system. We now turn to introduce the new invariant:

$$Q = \text{sign}\left[\text{pf}(H(\tau_x \otimes I_q))\right], \tag{75}$$

where τ_x is the first Pauli matrix, I is the identity matrix of dimension q (the dimension of the Hamiltonian is $2q$).

2.11 Calculating the angular momentum of the cooper pairs

The kinetic part of the single particle Hamiltonian (in first quantization):

$$\mathcal{H}_0 = \sum_{i,j} t_{i,j}|i\rangle\langle j| - \mu \sum_{i,j} |i\rangle\langle i|. \tag{76}$$

The position operator:

$$\mathbf{r} = \sum_i \mathbf{r}_i|i\rangle\langle i|. \tag{77}$$

The velocity operator:

$$\mathbf{v} = \sum_{i,j} v_{i,j}|i\rangle\langle j|, \quad v_{i,j} = \langle i|\mathbf{v}|j\rangle = \frac{1}{i\hbar}\langle i|\left[\mathbf{r}, \mathcal{H}_0\right]|j\rangle = -\frac{i}{\hbar}(\mathbf{r}_i - \mathbf{r}_j)(\mathcal{H}_0)_{ij}. \tag{78}$$

The eigenstates of the Hamiltonian are

$$|\psi_\epsilon\rangle = \sum_i \psi_{i,\epsilon}|i\rangle, \quad \psi_{i,\epsilon} = \langle i|\psi_\epsilon\rangle. \tag{79}$$

The angular momentum is

$$\begin{aligned}
\langle \mathbf{L}_z\rangle = \langle\psi_\epsilon|\mathbf{r}\times\mathbf{v}|\psi_\epsilon\rangle &= \sum_{i,j} \psi^*_{\epsilon,i}\psi_{\epsilon,j}\langle i|\mathbf{r}\times\mathbf{v}|j\rangle = \sum_{i,j}\psi^*_{\epsilon,i}\psi_{\epsilon,j}\left(\mathbf{r}_i \times \mathbf{v}_{i,j}\right) \\
&= -\frac{i}{\hbar}\sum_{i,j}\psi^*\epsilon, i\psi_{\epsilon,j}\left[x_i(y_i - y_j)\mathcal{H}_{i,j} - y_i(x_i - x_j)\mathcal{H}_{i,j}\right] \\
&= \frac{i}{\hbar}\sum_{i,j}\psi^*\epsilon, i\psi_{\epsilon,j}\mathcal{H}_{i,j}\left(x_i y_j - y_i x_j\right),
\end{aligned} \tag{80}$$

where for brevity we wrote $\mathcal{H}$ instead of $\mathcal{H}_0$. In the case that the angular momentum is calculated with respect to $\mathbf{r}_c = (x_c, y_c)$ we have:

$$\begin{aligned}\langle \mathbf{L}_z \rangle &= \langle \psi_\epsilon | (\mathbf{r} - \mathbf{r}_c) \times \mathbf{v} | \psi_\epsilon \rangle = \sum_{i,j} \psi^*_{\epsilon,i} \psi_{\epsilon,j} \langle i | (\mathbf{r} - \mathbf{r}_c) \times \mathbf{v} | j \rangle = \sum_{i,j} \psi^*_{\epsilon,i} \psi_{\epsilon,j} \left((\mathbf{r}_i - \mathbf{r}_c) \times \mathbf{v}_{i,j} \right) \\ &= -\frac{i}{\hbar} \sum_{i,j} \psi^* \epsilon, i \psi_{\epsilon,j} \left[(x_i - x_c)(y_i - y_j) \mathcal{H}_{i,j} - (y_i - y_c)(x_i - x_j) \mathcal{H}_{i,j} \right] \\ &= \frac{i}{\hbar} \sum_{i,j} \psi^* \epsilon, i \psi_{\epsilon,j} \mathcal{H}_{i,j} \left(x_i y_j - y_i x_j - x_c (y_i - y_j) + y_c (x_i - x_j) \right).\end{aligned} \tag{81}$$

3 Lattice gauge fields

In the presence of an external magnetic field the translation operators, which form the kinetic part of the Hamiltonian, are simply

$$T_x = \psi^\dagger_{m+1,n} \psi_{m,n} e^{i\theta^x_{m,n}} \qquad T_y = \psi^\dagger_{m,n+1} \psi_{m,n} e^{i\theta^y_{m,n}}. \tag{82}$$

The phase factors are defined as

$$\theta^x_{m,n} = \frac{e}{\hbar} \int_m^{m+1} A_x(x,n)\, \mathrm{d}x \qquad \theta^y_{m,n} = \frac{e}{\hbar} \int_n^{n+1} A_y(m,y)\, \mathrm{d}y \tag{83}$$

3.1 Calculating the the flux per plaquette

The number of flux quanta per plaquette ϕ_{mn} is related to the lattice curl of the phase factor:

$$\begin{aligned}\mathrm{rot}\theta_{m,n} = \Delta_x \theta^y_{m,n} - \Delta_y \theta^x_{m,n} &= \left(\theta^y_{m+1,n} - \theta^y_{m,n} - \theta^x_{m,n+1} + \theta^x_{m,n} \right) = \\ &= \frac{e}{\hbar} \int_{\substack{unit\\cell}} \mathbf{A} \cdot \mathrm{d}\mathbf{l} = 2\pi \frac{e}{h} \int \mathbf{B} \cdot \mathrm{d}\mathbf{s} = 2\pi \phi_{m,n}\end{aligned} \tag{84}$$

where the phase factors are defined as $\theta^x_{mn} = \frac{e}{\hbar} \int_m^{m+1} \mathbf{A} \cdot \mathrm{d}\mathbf{x}$ and $\theta^y_{mn} = \frac{e}{\hbar} \int_n^{n+1} \mathbf{A} \cdot \mathrm{d}\mathbf{y}$. Also, it is related to the accumulated phase of a single particle state, $|\psi\rangle = \psi_{i,j}|0\rangle$ surrounding a plaquette:

$$\begin{aligned}T^\dagger_y T^\dagger_x T_y T_x |\psi\rangle = T^\dagger_y T^\dagger_x T_y |i+1, j\rangle e^{i\theta^x_{i,j}} = T^\dagger_y T^\dagger_x |i+1, j+1\rangle e^{i\left(\theta^x_{i,j} + \theta^y_{i+1,j}\right)} = \\ T^\dagger_y |i, j+1\rangle e^{i\left(\theta^x_{i,j} + \theta^y_{i+1,j} - \theta^x_{i,j+1}\right)} = |i,j\rangle e^{i\left(\theta^x_{i,j} + \theta^y_{i+1,j} - \theta^x_{i,j+1} - \theta^y_{i,j}\right)} = |i,j\rangle e^{i2\pi\phi_{m,n}}\end{aligned} \tag{85}$$

3.2 The Landau gauge

Under this gauge, which is designed for a cylindrical lattice of width q, the vector potential is

$$A_x(x,y) = 0, \qquad A_y(x,y) = \frac{p\Phi_0 x}{qa}. \tag{86}$$

where p is the flux strength, q is the dimension of the lattice and $\Phi_0 = 2\pi$ is a flux quanta in natural units. The number of flux per plaquette, $\phi_{m,n}$ is given by

$$\phi_{m,n} = (m+1)\frac{p}{q} - m\frac{p}{q} = \frac{p}{q}. \tag{87}$$

The flux through a $q \times q$ lattice is $2\pi qp$ with $p \in \mathbb{Z}$.

3.3 The almost antisymmetric gauge

Under this gauge, which is designed for a toroidal lattice of size $q \times q+1$, the vector potential is

$$A_x(x, y) = \frac{p\Phi_0 y}{(q+1)a}, \qquad A_y(x, y) = \frac{p\Phi_0 x}{qa}. \tag{88}$$

where p is the flux strength and $\Phi_0 = 2\pi$ is a flux quanta in natural units. Under this gauge $\partial_y A_x$ is slightly greater than $\partial_x A_y$ and their contributions to the magnetic field are counter-oriented. The number of flux per plaquette, $\phi_{m,n}$ is given by

$$\phi_{m,n} = (m+1)\frac{p}{q} - m\frac{p}{q} - (n+1)\frac{p}{q+1} + n\frac{p}{q+1} = \frac{p}{q} - \frac{p}{q+1} = \frac{p}{q(q+1)}. \tag{89}$$

The flux through the entire lattice is $2\pi p$ with $p \in \mathbb{Z}$.

3.4 The construction of the Landau Gauge and the Almost Antisymmetric Gauge

In order to have an integer number of flux quanta flowing homogeneously through the entire lattice, we must have a constant rational number of flux quanta per plaquette. These requirements determines the form of the phase factors -

$$\mathrm{rot}(\theta_{i,j}) = 2\pi\phi_{i,j} = 2\pi\frac{p}{s} = \Delta_x\theta^y_{i,j} - \Delta_y\theta^x_{i,j} \Longrightarrow \theta^x_{i,j} = 2\pi\frac{p'}{s'}y, \quad \theta^y_{i,j} = 2\pi\frac{p''}{s''}x \tag{90}$$

with $s = s' \times s''$ being a divisor of the lattice size and $p = p'' \times s' - s'' \times p'$.

If we add a requirement for periodic boundary conditions over a rectangular lattice with $q_x \times q_y$ dimensions than

$$\theta^x_{i,q_y} = \theta^x_{i,0} - 2\pi n \Longrightarrow n = \frac{p'}{s'}q_y, \quad \theta^y_{q_x,j} = \theta^y_{0,j} - 2\pi m \Longrightarrow m = \frac{p''}{s''}q_x \quad \{m, n\} \in \mathbb{Z} \tag{91}$$

so s' equals to q_y or one of its divisors. The minimal flux per plaquette allowed by the gauge is determined by s. As the values of s' and s'' are larger the minimal value of the flux per plaquette is smaller. Thus, we choose $s' = q_y$ (which forces $n = p'$), likewise $s'' = q_x$. By setting $p' = 0$ and $q_x = q_y$, the Landau gauge is obtained. Also, setting $p' = p''$ and $q_y = q_x + 1$ would yield the almost anti-symmetric gauge. In both of the gauges there is a single parameter, p'' that controls the flux strength.

3.5 The Singular Gauge

In order to add "point flux" of a half flux quanta we can use the vector potential

$$\mathbf{A}(\mathbf{r}-\mathbf{r}_0) = \frac{\Phi_0}{4\pi}\boldsymbol{\nabla}\mathrm{Arg}(\mathbf{r}-\mathbf{r_0}) \tag{92}$$

where $\mathbf{r}_0$ denotes the "point flux" position on the lattice.

3.6 Magnetic unit cells and the degeneracy in the energy levels of Hofstadter Hamiltonian

The Hofstadter Hamiltonian describes a tight binding model for spinless electrons on a square lattice, in the presence of an external magnetic field. It can simplify be written as

$$\mathcal{H} = T_x + T_y + h.c\,, \tag{93}$$

where T_x and T_y are the translation operators. Calculating the commutation relation of the translation operators, when we act on a single-particle state (m,n), $|\psi_{mn}\rangle = \psi^\dagger_{m,n}|0\rangle$ yields

$$\begin{aligned}
T_xT_y|\psi_{i,j}\rangle &= \sum_{m,n} T_x\psi^\dagger_{m,n+1}\psi_{m,n}e^{i\theta^y_{m,n}}\psi^\dagger_{i,j}|0\rangle \\
&= \sum_{m,n}\psi^\dagger_{m+1,n}\psi_{m,n}\psi^\dagger_{i,j+1}|0\rangle e^{i\theta^y_{i,j}}e^{i\theta_{m,n}} = \psi^\dagger_{i+1,j+1}|0\rangle e^{i\left(\theta^y_{i,j}\theta^x_{i,j+1}\right)} \\
T_yT_x|\psi_{i,j}\rangle &= \sum_{m,n} T_y\psi^\dagger_{m+1,n}\psi_{m,n}e^{i\theta^x_{m,n}}\psi^\dagger_{i,j}|0\rangle = \\
&\sum_{m,n}\psi^\dagger_{m,n+1}\psi_{m,n}\psi^\dagger_{i+1,j}|0\rangle e^{\theta^x_{i,j}}e^{\theta^y_{m,n}} = \psi^\dagger_{i+1,j+1}|0\rangle e^{\left(\theta^x_{i,j}+\theta^y_{i+1,j}\right)} \\
\Rightarrow T_xT_y &= T_yT_xe^{\left(\theta^y_{ij}+\theta^x_{i,j+1}-\theta^x_{i,j}-\theta^y_{i+1,j}\right)} = T_yT_xe^{i\left(\Delta_x\theta^y_{i,j}-\Delta_y\theta^x_{i,j}\right)} = T_yT_xe^{i2\pi\phi_{i,j}}
\end{aligned} \tag{94}$$

The translation operators don't commute with each other and nor with the Hamiltonian. However, we can define magnetic translation operators that do commute with the Hamiltonian as follows:

$$\tilde{T}_x = \psi^\dagger_{m+1,n}\psi_{m,n}e^{i\chi^x_{m,n}} \qquad \tilde{T}_y = \psi^\dagger_{m,n+1}\psi_{m,n}e^{i\chi^y_{m,n}}. \tag{95}$$

In order to determine the phase $\chi^x_{m,n}$ we demand that a) the magnetic translation operator obey the commutation relation $\left[T_x,\tilde{T}_x\right]=0$ and b) $\left[T_y,\tilde{T}_x\right]=0$. Starting from the first requirement:

$$\begin{aligned}
T_x\tilde{T}_x|\psi_{i,j}\rangle &= T_x\sum_{m,n}\psi^\dagger_{m+1,n}\psi_{m,n}\psi^\dagger_{i,j}|0\rangle e^{i\chi^x_{m,n}} = \sum_{m,n}\psi^\dagger_{m+1,n}\psi_{m,n}\psi^\dagger_{i+1,j}|0\rangle e^{i\left(\chi^x_{i,j}+\theta^x_{m,n}\right)} \\
&= \psi^\dagger_{i+2,j}|0\rangle e^{i\left(\chi^x_{i,j}+\theta^x_{i+1,j}\right)} \\
\tilde{T}_xT_x|\psi_{i,j}\rangle &= \psi^\dagger_{i+2,j}|0\rangle e^{i\left(\theta^x_{i,j}+\chi^x_{i+1,j}\right)} \\
&\Rightarrow \left[T_x,\tilde{T}_x\right] = \psi^\dagger_{m+2,n}\psi_{m,n}e^{i\left(\theta^x_{m+1,n}+\chi^x_{m,n}\right)}\left(1-e^{i\left(\chi^x_{m+1,n}+\theta^x_{m,n}-\theta^x_{m+1,n}-\chi^x_{m,n}\right)}\right)
\end{aligned} \tag{96}$$

Thus, we deduce that the constrain which maintains the commutativity of T_x and $\tilde{T}_x$ is

$$\Delta_x \chi^x_{i,j} = \Delta_x \theta^x_{i,j} \tag{97}$$

Continuing with the second requirement:

$$\begin{aligned} T_x \tilde{T}_y |\psi_{ij}\rangle &= T_x \sum_{m,n} \psi^\dagger_{m,n+1}\psi_{m,n}\psi^\dagger_{i,j}|0\rangle e^{i\chi^y_{m,n}} = \sum_{m,n} \psi^\dagger_{m+1,n}\psi_{m,n}\psi^\dagger_{i,j+1}|0\rangle e^{i\left(\theta^x_{m,n}+\chi^y_{i,j}\right)} \\ &= \psi^\dagger_{i+1,j+1}|0\rangle e^{i\left(\theta^x_{i,j+1}+\chi^y_{i,j}\right)} \\ \tilde{T}_y T_x |\psi_{ij}\rangle &= \tilde{T}_y \sum_{m,n} \psi^\dagger_{m+1,n}\psi_{m,n}\psi^\dagger_{i,j}|0\rangle e^{i\theta^x_{m,n}} = \sum_{m,n} \psi^\dagger_{m,n+1}\psi_{m,n}\psi^\dagger_{i+1,j}|0\rangle e^{i\left(\chi^y_{m,n}+\theta^x_{i,j}\right)} \\ &= \psi^\dagger_{i+1,j+1}|0\rangle e^{i\left(\theta^x_{i,j}+\chi^y_{i+1,j}\right)} \\ \Rightarrow [T_x, \tilde{T}_y] &= \psi^\dagger_{m+1,n+1}\psi_{m,n}\left(1 - e^{i\left(\theta^x_{m,n}+\chi^y_{m+1,n}-\theta^x_{m,n+1}-\chi^y_{m,n}\right)}\right) e^{i\left(\theta^x_{m,n+1}+\chi^y_{m,n}\right)} \end{aligned} \tag{98}$$

From here we deduce that the second constrain which assures the commutativity of T_y and $\tilde{T}_x$ is

$$\Delta_x \chi^y_{i,j} = \Delta_y \theta^x_{i,j} \tag{99}$$

Using the relation $2\pi\phi_{ij} = \Delta_x\theta^y_{ij} - \Delta_y\theta^x_{ij}$, the two conditions can be represented as

$$\Delta_x \chi^x_{i,j} = \Delta_x \theta^x_{i,j}, \qquad \Delta_y \theta^x_{i,j} + 2\pi\phi_{i,j}. \tag{100}$$

In the case of constant flux per plaquette the solution is

$$\chi^x_{i,j} = \theta^x_{i,j} + 2\pi j \phi_{i,j} \tag{101}$$

Similarly,

$$\left[T_x, \tilde{T}_x\right] = 0 \Longrightarrow \Delta_x \chi^x_{i,j} = \Delta_x \theta^x_{i,j}, \qquad \left[T_y, \tilde{T}_x\right] = 0 \Longrightarrow \Delta_x \chi^y_{i,j} = \Delta_y \theta^x_{i,j} \tag{102}$$

and the solution of the conditions for $\chi^y_{i,j}$ is

$$\chi^y_{i,j} = \theta^y_{i,j} - 2\pi i \phi_{i,j} \tag{103}$$

Also, the condition for the magnetic translation operators $\tilde{T}_x$ and $\tilde{T}_y$ to commute, likewise eq.(94), is

$$\Delta_x \chi^y_{i,j} - \Delta_y \chi^x_{i,j} = \Delta_x \theta^y_{i,j} - \Delta_y \theta^x_{i,j} = 2\pi\phi_{i,j} = 2\pi p. \tag{104}$$

with $p \in \mathbb{Z}$. Thus, except for the case that $\phi_{i,j}$ is an integer they do not commute. This exception is not a great help since it means that the flux through the whole lattice is a flux quanta multiplied by its size. On the other hand, if we require the flux number per plaquette to be rational,

$$\phi_{i,j} = \frac{p}{s}, \quad \{p, s\} \in \mathbb{Z} \tag{105}$$

than

$$\left[\tilde{T}_x^{s_x}, \tilde{T}_y^{s_y}\right] = 0, \text{ where } s_x \times s_y = s \tag{106}$$

If the momentum states, $|\mathbf{k}\rangle$ are also eigenstates of the magnetic translation operators, $\tilde{T}_x^{s_x}$ and $\tilde{T}_y^{s_y}$ than they define a magnetic cell, in which the Hamiltonian is periodic.

Under the Landau gauge, for a square lattice ($q \times q$) the phases of the magnetic translation operators are

$$\chi_x = \theta_{ij}^x + 2\pi j\phi = 2\pi j\frac{p}{q}, \quad \chi_y = \theta_{ij}^y - 2\pi i\phi = 2\pi i\left(\frac{p}{q} - \frac{p}{q}\right) = 0 \tag{107}$$

Thus, the magnetic translation operators satisfy

$$\tilde{T}_x^q|\mathbf{k}\rangle = e^{ik_xq}|\mathbf{k}\rangle, \quad \tilde{T}_y|\mathbf{k}\rangle = e^{ik_y}|\mathbf{k}\rangle. \tag{108}$$

The first is obtained by noticing that for a single particle - $\tilde{T}_x^q|\psi\rangle = \psi_{m+q,n}^\dagger\psi_{m,n}|\psi\rangle$ with $|\psi\rangle = \psi_{m,n}^\dagger|0\rangle$ and representing the operator in momentum space. The second is a straight forward representation in k space. These operators, as shown above, also commute,

$$\left[\tilde{T}_x^q, \tilde{T}_y\right] = 0. \tag{109}$$

The Bloch conditions, $[H, \tilde{T}_x] = 0$ and $[H, \tilde{T}_y] = 0$ imply that $H|\mathbf{k}\rangle = E(\mathbf{k})|\mathbf{k}\rangle$. From here we can prove that the Landau-level problem on a lattice has a q-fold degeneracy at different wavevectors -

$$\tilde{T}_y\tilde{T}_x|k_x, k_y\rangle = e^{-i2\pi\phi}\tilde{T}_x\tilde{T}_y|k_x, k_y\rangle = e^{i(k_y-2\pi\phi)}\tilde{T}_x|k_x, k_y\rangle \Rightarrow \tilde{T}_x|k_x, k_y\rangle = |k_x, k_y - 2\pi\phi\rangle \tag{110}$$

The eigenstates $|k_x, k_y - 2\pi\phi\rangle$ and $|k_x, k_y\rangle$ have identical energy, since $\tilde{T}_x$ commutes with the Hamiltonian. Because the flux number, $\phi = \frac{p}{q}$ is rational, the spectrum is q-fold degenerate, corresponding to the application of $\tilde{T}_x$ q times.

4 The geometric phases of the p-wave superconductor ground state as two vortices are exchanged

In this section we derive a formula for the Berry phase of a spinless chiral p-wave superconductor ground state due to the simultaneous exchange of two vortex defects. This formula is suited for the case that the underling structure of the p-wave superconductor is a two-dimensional tight-binding square lattice with Dirichlet boundary conditions.

4.1 Derivation of the Gauge-Independent Berry Phase formula

We introduce an instantaneous orthonormal eigenstates $|n(\mathbf{R})\rangle$ of the Hamiltonian $\mathcal{H}(\mathbf{R})$ at each point $\mathbf{R}$:

$$\mathcal{H}(\mathbf{R})|n(\mathbf{R})\rangle = E_n(\mathbf{R})|n(\mathbf{R})\rangle \tag{111}$$

The time evolution of the system is given by

$$\mathcal{H}(\mathbf{R}(t))|\psi(t)\rangle = i\hbar\partial_t|\psi(t)\rangle, \text{ where } |\psi(t)\rangle = e^{i\theta(t)}|n(\mathbf{R}(t))\rangle. \tag{112}$$

Substituting the time dependent wave function yields the differential equation

$$E_n(\mathbf{R}(t))|n(R(t))\rangle = -\hbar\partial_t\theta(t)|n(\mathbf{R}(t))\rangle + i\hbar\partial_t|n(\mathbf{R}(t))\rangle \tag{113}$$

which has a solution

$$\theta(t) = \underbrace{i\int_{t_i}^{t_f} \mathrm{d}t\langle n(\mathbf{R}(t))|\partial_t|n(\mathbf{R}(t))\rangle}_{\text{geometric phase}} - \underbrace{\frac{1}{\hbar}\int_{t_i}^{t_f} \mathrm{d}t\ E_n(\mathbf{R}(t))}_{\text{dynamical phase}}. \tag{114}$$

We are interested only in the geometric phase, known also as the Berry phase. The integration over t can be regarded as some parametrization of $\mathbf{R}$ so we rewrite it as a contour integral,

$$\gamma_n = i\int_{t_i}^{t_f} \mathrm{d}t\ \langle n(\mathbf{R}(t'))|\nabla_{\mathbf{R}}|n(\mathbf{R}(t'))\rangle\dot{\mathbf{R}} = i\oint_C d\mathbf{R}\langle n(\mathbf{R})|\nabla_{\mathbf{R}}|n(\mathbf{R})\rangle. \tag{115}$$

Further more we can write the Berry phase as $\gamma_n = \oint_C d\mathbf{R}\cdot\mathbf{A}_n(\mathbf{R})$, where $\mathbf{A}_n(\mathbf{R}) = i\langle n(\mathbf{R})|\nabla_{\mathbf{R}}|n(\mathbf{R})\rangle = -\Im\langle n(\mathbf{R})|\nabla_{\mathbf{R}}|n(\mathbf{R})\rangle$ is defined as the Berry vector potential. The last equality is based on the fact that

$$\begin{aligned}
\langle n(\mathbf{R})|n(\mathbf{R})\rangle = 1 \quad &\Rightarrow \quad \underbrace{\langle n(\mathbf{R})|\nabla_{\mathbf{R}}|n(\mathbf{R})\rangle}_{\eta} + \underbrace{\langle n(\mathbf{R})|\nabla_{\mathbf{R}}|n(\mathbf{R})\rangle^*}_{\eta^*} = 0 \\
&\Rightarrow \quad 2\Re(\eta) + i(\Im(\eta) - \Im(\eta)) = 0 \quad \Rightarrow \quad \Re(\eta) = 0 \\
&\Rightarrow \quad \Re(\langle n(\mathbf{R})|\nabla_{\mathbf{R}}|n(\mathbf{R})\rangle) = 0
\end{aligned}$$

The time independent wave function, $|n(\mathbf{R})\rangle$ is defined uniquely up to a global phase which can be gauged. Under the gauge transformation $|n(\mathbf{R})\rangle \to e^{i\zeta(\mathbf{R})}|n(\mathbf{R})\rangle$, where $\zeta(\mathbf{R})$ must maintain the smoothness and the single-valueness of the wave function.[2] Consequentially, the Berry vector potential $\mathbf{A}_n(\mathbf{R})$ is transformed as $A_n(\mathbf{R}) \to A_n(\mathbf{R}) - \nabla_{\mathbf{R}}\zeta(\mathbf{R})$. The Berry phase will defer by $\Delta\gamma_n = -\oint_C \mathrm{d}\mathbf{R}\nabla_{\mathbf{R}}\zeta(\mathbf{R}) = \zeta(\mathbf{R}(t_i)) - \zeta(\mathbf{R}(t_f)) = 2\pi m$ with m being an integer. The last equality is a result of $\mathbf{R}(t_i)$ and $\mathbf{R}(t_f)$ referring to the same point in the parameter space while $\zeta(\mathbf{R})$ is allowed to be multivalued as long as the wave function is kept single-valued.

In order to describe the simultaneous exchange of two vortices, we need no more than a three-dimensional parameter space. For a 3D closed path C, the Berry phase is a gauge-invariant quantity as can easily seen by applying the Stokes

[2] The $U(1)$ transformation changes the Hamiltonian as $(\overbrace{U^{-1}HU}^{H'})(\overbrace{U^{-1}|n\rangle}^{|n'\rangle}) = E(\overbrace{U^{-1}|n\rangle}^{|n\rangle'})$ with $U = e^{-i\zeta(\mathbf{R})}$.

theorem:

$$\gamma_n = -\Im \iint_S \mathrm{d}\mathbf{S} \cdot (\boldsymbol{\nabla}_\mathbf{R} \times \langle n(\mathbf{R})|\boldsymbol{\nabla}_\mathbf{R}|n(\mathbf{R})\rangle) = -\Im \iint_S \mathrm{d}S_i \epsilon_{ijk} \partial_j \underbrace{(\langle n(\mathbf{R})|\boldsymbol{\nabla}_\mathbf{R}|n(\mathbf{R})\rangle)_k}_{\langle n|\partial_k|n\rangle} \\ = -\Im \iint_S \mathrm{d}\mathbf{S} \cdot \langle \boldsymbol{\nabla}_\mathbf{R} n| \times |\boldsymbol{\nabla}_\mathbf{R} n\rangle \tag{116}$$

The last equality lies on the fact that the eigenstates are single-valued and smooth functions of the parameter space, **R**, so

$$\epsilon_{ijk}\partial_j (\langle n(\mathbf{R})|\partial_k|n(\mathbf{R})\rangle) = \epsilon_{ijk}\langle \partial_j n(\mathbf{R})|\partial_k n(\mathbf{R})\rangle + \epsilon_{ijk}\langle n(\mathbf{R})|\partial_j\partial_k n(\mathbf{R})\rangle = \epsilon_{ijk}\langle \partial_j n(\mathbf{R})|\partial_k n(\mathbf{R})\rangle. \tag{117}$$

In a numerical diagonalization of the Hamiltonian to obtain the eigenstates at each **R**, the diagonalization procedure will output states with wildly different phase factors, thereby preventing the taking of derivatives. We must gauge-smoothen first, but this is a nontrivial procedure. However, we overcome this difficulty by rewriting the formula using only projectors and gradients of them, which are essentially gauge independent. We start with inserting a complete set of eigenstates $\sum_m |m\rangle\langle m| = 1$ between the two states gradient,

$$\epsilon_{ijk}\langle \partial_j n|\partial_k n\rangle = \epsilon_{ijk} \underbrace{\langle \partial_j n|n\rangle}_{Imaginary} \underbrace{\langle n|\partial_k n\rangle}_{Imaginary} + \sum_{m\neq n} \epsilon_{ijk}\langle \partial_j n|m\rangle\langle m|\partial_k n)\rangle \tag{118}$$

The first term term is real, giving no contribution to the imaginary part that is used to compute the Berry phase. Hence, we have

$$\gamma = \Im \iint_S \mathrm{d}S_i \sum_{m\neq n} \epsilon_{ijk}\langle \partial_j n|m\rangle\langle m|\partial_k n\rangle. \tag{119}$$

Our next step is to move the derivatives from the eigenstates to the Hamiltonian, that can always be represented using projectors and as so is gauge independent. Noticing that

$$E_n\langle m|\boldsymbol{\nabla}_\mathbf{R} n\rangle = \langle m|\boldsymbol{\nabla}_\mathbf{R}(\mathcal{H}|n\rangle) = \langle m|(\boldsymbol{\nabla}_\mathbf{R}\mathcal{H})|n\rangle + \langle m|\mathcal{H}|\boldsymbol{\nabla}_\mathbf{R} n\rangle \tag{120}$$

we can write the identity

$$\langle m|\boldsymbol{\nabla}_\mathbf{R} n\rangle = \frac{\langle m|\boldsymbol{\nabla}_\mathbf{R}\mathcal{H}|n\rangle}{E_n - E_m} \tag{121}$$

and a similarly

$$\langle \boldsymbol{\nabla}_\mathbf{R} n|m\rangle = \langle m|\boldsymbol{\nabla}_\mathbf{R} n\rangle^* = \frac{\langle n|\boldsymbol{\nabla}_\mathbf{R}\mathcal{H}|m\rangle}{E_n - E_m}. \tag{122}$$

Finally the single particle gauge independent Berry phase formula is

$$\gamma_n = -\Im \iint_S \mathrm{d}\mathbf{S} \cdot \sum_{m\neq n} \frac{\langle n|\boldsymbol{\nabla}_\mathbf{R}\mathcal{H}|m\rangle \times \langle m|\boldsymbol{\nabla}_\mathbf{R}\mathcal{H}|n\rangle}{(E_n - E_m)^2} \tag{123}$$

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צעד ראשון לקראת איכסון וביצוע מניפולציות במידע קוונטי המקודד במרחב מצבי היסוד אשר נפרש על ידי מיורנות המאוכלסות במערבולות מסוג אבריקוסוב הוא לאפיין את התכונות האלקטרוניות שלהן. לקראת מטרה זו אנו מציגים מערך כלים הכולל: (א) מבנה כללי לפאזה של פרמטר סדר מרוכב המסוגל לקודד כל קונפיגורציה מרחבית של מערבולות על גביי טורוס או צילנדר. (ב) כיול של וקטור הפוטנציאל אשר אנו מכנים ״הכיול הכמעט אנטי־סימטרי״, המאפשר להגיע לרזולוציה הגבוהה ביותר של שדה מגנטי הומוגני במערכת עם תנאי שפה מחזוריים. אנו משתמשים במערך הכלים יחד עם משפט בלוך כדי לפתור המילטוניאן צמוד־קשר המתאר סריג מערבולות אינסופי במוליך־על כיראלי, דו מימדי מסוג $p + ip$. אנו מוצאים כי נוכחות מצבים קשורים יוצרת השפעה חזקה על מוליכות הול וכתוצאה מכך גם על אפקט קר, המכומת ע״י זווית קר שפרופורציונלית למוליכות הול.

מילות מפתח: מולכי־על טופולוגיים, מולכי־על כיראליים, צומת ג׳וזפסון מעגלית, תורת שדות אפיקטיבית, איבר צ׳רן־סיימונס, מערבולות אבריקוסוב, פאזה אבלית, ספין טופולוגי.

תקציר

עם התגברות העניין במחשבים קוונטים, שיטות חדשות למימושם נדרשות. מערבולות בנוזלי־על כיראליים, חסרי ספין ודו־מימדיים מאוכלסות על ידי מיורנות ממוקמות באנרגיה אפס (או בפשטות ־ מיורנות) שמעניקות להן סטטיסטיקת החלפות לא אבלית. תכונה זו מאפשרת ביצוע טרנספורמציות אוניטריות בתוך מרחב מצבי היסוד ולכן המערכת הפיזיקלית יכולה לשמש לבניית מחשבים קוונטים העמידים לשגיאות. מונעים מהפוטנציאל הטמון במערבולות לעיבוד מידע קוונטי טופולוגי, פיתחנו תאוריה אפיקטיבית לאנרגיות נמוכות עבור מערבולות בנוזלי־על מסוג $p+ip$. בפיתוח השתמשנו בטרנספורמצית כיול חד־ערכית שמשמרת את סימטרית החלקיק־חור בפעולה. התאוריה כוללת את הפיסיקה של דינמיקת המערבולות, כגון כח מגנוס שפרופורציונלי לצפיפות האלקטרונית בנוזלי־העל. כמו כן, התאוריה מכילה שני אברי צ'רן סיימונס ־ שלם וחלקי. הראשון מנבא פאזה אבלית אוניברסלית, $\exp(i\pi/8)$, בעת החלפת זוג מערבולות. עם זאת, לפאזה זו יש תיקונים לא־אוניברסלים שממוסכים בנוזלי־על טעונים ומיוחסים לאיבר הצ'רן־סיימונס החלקי.

יש מספר סוגים שונים של מערכות בהם החלפת מיורנות יכולה להתבצע בפועל. אחת מהמערכות הרלוונטיות היא צומת ג'וזפסון טופולוגית מאחר ומבחינה ניסויית ניתן לשלוט בתנועת המערבולות בפשטות. הוכח בעבר כי מערבולות ג'וזפסון בצומת ג'וזפסון טופולוגית אכן מאכלסות מיורנות ובתיזה זו אנו מראים שיש להן סטטיסטיקת החלפות הזהה לזו של המערבולות בתוך החומר. הדבר נעשה על ידי הצעת המילטוניאן שמתאר את התנועה הקולקטיבית של פאזת סוליטון הכלוא בצומת ג'וזפסון מעגלית העשויה משני מוליכי־על טופולוגיים. פיתחנו את משוואות התנועה הקוונטית של הסוליטון הכלוא, ומתוכן אנו מחשבים את הפאזה הגאומטרית שנצברת כאשר הוא מקיף את הצומת ומראים כי ניתן להסיק אותה מהמתח על הצומת.

העבודה נעשתה בהדרכת

פרופ' איתן גרוספלד

במחלקה לפיסיקה

בפקולטה למדעי הטבע

אוניברסיטת בן גוריון בנגב

פאזות גאומטריות במוליכי־על טופולוגיים

מחקר לשם מילוי חלקי של הדרישות לקבלת תואר

"דוקטור לפילוסופיה"

מאת

דניאל אריעד

הוגש לסינאט אוניברסיטת בן גוריון בנגב

אישור המנחה: ____________________

אישור דיקן בית הספר ללימודי מחקר מתקדמים ע"ש קרייטמן: ____________________

י"ב ניסן ה'תשע"ח   28 למרץ 2018

באר־שבע, ישראל

פאזות גאומטריות במוליכי־על טופולוגיים

מחקר לשם מילוי חלקי של הדרישות לקבלת תואר

"דוקטור לפילוסופיה"

מאת

דניאל אריעד

הוגש לסינאט אוניברסיטת בן גוריון בנגב

י"ב ניסן ה'תשע"ח　　　　28 למרץ 2018

באר־שבע, ישראל